Mathematics and Computer Science
2016; 1(3): 56-60
http://www.sciencepublishinggroup.com/j/mcs
doi: 10.11648/j.mcs.20160103.14

On *A*-Self-Adjoint, *A*-Unitary Operators and Quasiaffinities

Isaiah N. Sitati*, Bernard M. Nzimbi, Stephen W. Luketero, Jairus M. Khalagai

School of Mathematics, College of Biological and Physical Sciences, University of Nairobi, Nairobi, Kenya

Email address:
sitatiisaiah@gmail.com (I. N. Sitati)
*Corresponding author

To cite this article:
Isaiah N. Sitati, Bernard M. Nzimbi, Stephen W. Luketero, Jairus M. Khalagai. On *A*-Self-Adjoint, *A*-Unitary Operators and Quasiaffinities. *Mathematics and Computer Science.* Vol. 1, No. 3, 2016, pp. 56-60. doi: 10.11648/j.mcs.20160103.14

Received: August 8, 2016; **Accepted**: August 18, 2016; **Published**: September 7, 2016

Abstract: In this paper, we investigate properties of *A-self-adjoint* operators and other relations on Hilbert spaces. In this context, A is a self-adjoint and an invertible operator. More results on operator equivalences including similarity, unitary and metric equivalences are discussed. We also investigate conditions under which these classes of operators are self- adjoint and unitary. We finally locate their spectra.

Keywords: *A*-Self-Adjoint, *A*-Unitary, Hilbert Space, Metric Equivalence, Quasiaffinities

1. Introduction

Throughout this paper Hilbert spaces or subspaces will be denoted by capital letters, H and K respectively and T, A, B etc denote bounded linear operators where an operator means a bounded linear transformation. $B(H)$ will denote the Banach algebra of bounded linear operators on H. $B(H, K)$ denotes the set of bounded linear transformations from H to K, which is equipped with the (induced uniform) norm.

If $T \in B(H)$, then T^* denotes the adjoint while $Ker(T)$, $Ran(T)$, $\overline{M}$ and $M^{\perp}$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H respectively. For an operator T, we also denote by $\sigma(T)$, $\| T \|$ the spectrum and norm of T respectively.

A contraction on H is an operator $T \in B(H)$ such that $T^*T \leq I$ (i.e. $\| Tx \| \leq \| x \|$ $\forall x \in H$). A *strict or proper* contraction is an operator T with $T^*T < I$ (i.e. $\sup_{0 \neq x} \frac{\|Tx\|}{\|x\|} < 1$). If $T^*T = I$, then T is *called a non-strict contraction* (or an *isometry*). Many authors like Kubrusly [7] have extensively studied this class of operators.

An operator $T \in B(H)$ is said to be positive if $\langle Tx, x \rangle \geq 0$ $\forall x \in H$. Suppose that $A \in B(H)$ is a positive operator, then an operator $T \in B(H)$ is called an $A-contraction$ on H if $T^*AT \leq A$. If equality holds, that is $T^*AT = A$, then T is called an $A-isometry$, where A is a self adjoint and invertible operator.

In this research, we put more conditions on A. In particular, if A is a self adjoint and invertible operator, then we call such an $A-isometry$ an $A-Unitary$. Let T be a linear operator on a Hilbert space H.

We define the $A-adjoint$ of T to be an operator S such that $AS = T^*A$ .The existence of such an operator is not guaranteed. It may or may not exist. In fact a given $T \in B(H)$ may admit many $A-adjoints$ and if such an $A-adjoint$ of T exists, we denote it as $T^{[*]}$. Thus $AT^{[*]} = T^*A$ .We are making an assumption that A is invertible and so $T^{[*]} = A^{-1}T^*A$. It is also clear that $A-adjoint$ of T is the adjoint of T if $T = I$. By [2], T admits an $A-adjoint$ if and only if $Ran(T^*A) \subset Ran(A)$. In this case the operator A is acting as a signature operator on H.

Two operators $T \in B(H)$ *and* $S \in B(K)$ *are similar (denoted* $T \approx S$) if there exists an operator $X \in \mathcal{G}(H, K)$ where $\mathcal{G}(H, K)$ is a Banach subalgebra of $B(H, K)$ which is an invertible operator from H to K such that $XT = SX$ (i.e, $X^{-1}SX$ or $S = XTX^{-1}$). $T \in B(H)$ and $S \in B(K)$ are *unitarily equivalent* (denoted$T \cong S$), if there exists a unitary operator $U \in \mathcal{G}(H, K)$ such that $UT = SU (i.e, T = U^*SU$ or $equivalently$ $S = UTU^*)$.

Two operators are considered the “same” if they are unitarily equivalent since they have the same, properties of invertibility, normality, spectral picture (norm, spectrum and spectral radius).

An operator $X \in B(H, K)$ is *quasi-invertible* or a *quasi-affinity* if it is an injective operator with dense range (i.e. $Ker\ X = \{0\}$ and $\overline{Ran(X)} = K$; equivalently, $Ker\ X = \{\bar{0}\}$

and, $Ker\ X^* = \{\bar{0}\}$ thus $X \in B\ (H,K)$ is quasi-invertible if and only if $X^* \in B(K,H)$ is quasi-invertible).

An operator $T \in B(H)$ is a *quasi-affine transform* of $S \in B(K)$ if there exists a quasi-invertible $X \in B\ (H,K)$ such that $XT = SX$ $(ie\ X\ intertwines\ T\ and\ S)$.$T$ is a *quasiafiine transform* of S if there exists a quasinvertible operator intertwining T to S.

Two operators $A, B \in B(H)$ are said to be *almost similar (a.s) (denoted by* $A \overset{a.s}{\sim} B$) if there exists an invertible operator N such that the following two conditions are satisfied:$A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$.

Two operators $A, B \in B(H)$ are said to be *metrically equivalent (denoted by* $A \overset{m.e}{\sim} B$) if $\| Ax \| = \| Bx \|$ (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in H$) or $A \overset{m.e}{\sim} B$ if $A^*A = B^*B$ .This concept was introduced by Nzimbi et al ([8]).

Two linear operators $T \in B(H)$ and $S \in B(K)$ are said to be $A - unitarily$ equivalent (denoted$T \cong S$), if there exists an $A - unitary$ operator $U \in \mathcal{G}\ (H,K)$ such that $TU = US$.

We shall also define the following classes of operators in this paper:

An operator $T \in B(H)$ is said to be an *involution if* $T^2 = I$.

An operator $T \in B(H)$ is said to be *self-adjoint or Hermitian if* $T^* = T$ *(equivalently, if* $\langle Tx, x\rangle\ \forall \in H$).

An operator $T \in B(H)$ is said to be *unitary if* $T^*T = TT^* = I$ and *normal if* $T^*T = TT^*$*(equivalently, if* $\| Tx \| = \| T^*x \|$ $\forall\ x\ \in H$).

An operator $T \in B(H)$ is said to be *a partial isometry* if $T = TT^*T$ or equivalently, if T^*T is a projection.

An operator $T \in B(H)$ is said to be *quasinormal* if $T(T^*T) = (T^*T)T$ or equivalently if T commutes with (T^*T) that is $[T, T^*T] = 0$.

Let H and K be Hilbert spaces. An operator $X \in B(H,K)$ is invertible if it is injective (one -to- one) and surjective (onto or has dense range); equivalently if $Ker(X) = \{0\}$ and $\overline{Ran(X)} = K$. We denote the class of invertible linear operators by $\mathcal{G}\ (H,K)$.The commutator of two operators A and B, denoted by $[A,B]$ is defined by $AB - BA$. The self−commutator of an operator A is $[A, A^*] = A^*A - AA^*$.

Suppose $A \in B(H)$ is a self-adjoint and invertible operator, not necessarily unique. An operator $T \in B(H)$ is said to be $A - self\ adjoint\ \ if\ \ T^* = ATA^{-1}$ (equivalently, $T^{[*]} = T$) , $A - skew\ adjoint\ \ \ if\ \ T^* = -ATA^{-1}$ (equivalently, $T^{[*]} = -T$) , $A - normal$ if $A^{-1}\,T^*AT = TA^{-1}\,T^*A$ or equivalently, $T^{[*]}\,\mathrm{T} = TT^{[*]}, A - unitary$ if $T^*AT = A$ or equivalently, $T^{[*]} = T^{-1}$.Clearly, an A-isometry whose range is dense in H is an $A -\ unitary$.

2. Basic Results

We shall investigate operators in a Hilbert Space H that are not self-adjoint. It is well known that every self- adjoint operator has a real spectrum.

The following results will form a basis for our discussion throughout this paper.

Theorem 2.1 [7, Theorem 2.1]. *An invertible operator T is a product of two self-adjoint operators if and only if* $\sigma(T) = \sigma(T^*)$.

Proof: [See 7].

Remark: The product of two self-adjoint operators need not to have real spectrum. To justify our claim, we consider self-adjoint operators $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The product $PQ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has a purely imaginary spectrum$\{i, -i\}$. Denoting by $\mathfrak{I}_0$ the set of all invertible products of self-adjoint operators P and Q and by $\mathfrak{I}$ the set of invertible operators that are similar to their adjoints, we see that $\mathfrak{I}_0 \subseteq \mathfrak{I}$. The above theorem asserts that $\mathfrak{I} \subseteq \mathfrak{I}_0$ is also valid. By using the invariance of these two classes under similarity transformations, we notice that $\mathfrak{I}$ is strictly larger than the class of operators that are similar to their adjoints. We can give an example of a unilateral shift operator on $H = l^2$ in this context.

Theorem 2.2 [12]:$T \in B(H)$ *is unitarily equivalent to its adjoint if and only if T is a product of a symmetry (self-adjoint or unitary involution) and a self-adjoint operator.*

Theorem 2.3 *[7, pp. 6]: Two normal operators that are similar are unitarily equival*ent.

Remark: Any invertible normal operator which is similar to its adjoint can be expressed as a product of self-adjoint operators, that is, if T is normal and $T \in \mathfrak{I}$, then $T \in \mathfrak{I}_0$.

Proposition 2.4 [17]: *If $T \in B(H)$ is self-adjoint and injective, then T^{-1} is also self-adjoint.*

Remark: Just like other bounded linear operators, the *A-self adjoint* operation satisfies the following properties which can easily be shown using the definition of an $A - self\ adjoint\ of\ T, that\ is,\ T^* = ATA^{-1}$:

(a). $(T_1 + T_2)^{[*]} = {T_1}^{[*]} + {T_2}^{[*]}$

(b). $(T_1T_2)^{[*]} = {T_2}^{[*]}{T_1}^{[*]}$

(c). $(T^{-1})^{[*]} = (T^{[*]})^{-1}$

(d). $(\alpha T)^{[*]} = \bar{\alpha}T^{[*]}$

3. *A*-Self-Adjoint Operators

Definition: A Jordan algebra $\mathbb{J}$ consists of a real vector space equipped with a bilinear product xy satisfying the commutative law and the Jordan identity: $xy = yx$ and $(x^2y)x = x^2(yx)\ \forall\ x, y \in \mathbb{R}$. A Jordan algebra is formally real if $\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow x_1 = \cdots = x_n = 0$.

Remark: An associative algebra, $\mathbb{J}'$ over a real Hilbert space H gives rise to a Jordan algebra $\mathbb{J}$ under quasi-multiplication: the product $xy = \frac{1}{2}(xy + yx)$ is commutative and satisfies the Jordan identity since $4(x^2y)x = (x^2y + yx^2)x + x(x^2y + yx^2) = x^2yx + yx^3 + x^3y + xyx^2 = x^2(yx + yx) + x^2(yx + yx)4x^2(yx)$.

We say that a Jordan algebra $\mathbb{J}'$ is $special$ if it can be realized as a Jordan subalgebra of some Jordan algebra $\mathbb{J}$.

Example: If $\mathbb{J}_A$ is a set of Jordan operators, then the subspace of hermitian operator ${T_1}^{[*]} = T_1$ is also closed under the Jordan product, since if ${T_1}^{[*]} = T_1$ and ${T_2}^{[*]} = T_2$, then $(T_1T_2)^{[*]} = {T_1}^{[*]}{T_2}^{[*]} = T_2T_1 = T_1T_2$ forms a special algebra $H(\mathbb{J}_A, [*])$. These hermitian algebras are the archetypes of all Jordan algebras. We can easily check that hermitian matrices over $\mathbb{R}\ or\ \mathbb{C}$ form special Jordan algebras that are formally real.

We shall investigate the Jordan algebra $\mathbb{J}_A$ of $A - self\ adjoint$ Operators denoted by the set $\mathbb{J}_A = \{T \in B(H): T^{[*]} = T\}$. Note that just like many other algebras like the Lie algebra $\mathbb{L}_A, \mathbb{J}_A$ is an $\mathbb{R}$- linear subspace. That is, it is closed under real linear combinations.

We outline in the following results some conditions that guarantee an $A - self\ adjoint$ to be self-adjoint.

Proposition 3.1: [7]. *Every self –adjoint operator T is* $A - self\ adjoint$.

Remark: The converse of the above proposition is not generally true. For consider the operators $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A quick calculation reveals that T is $A - self\ adjoint$ but it is not self-adjoint. We note that $A - self\ adjointness$ coincides with self-adjointness when A is an identity operator.

We now answer the question: when is an $A - self\ adjoint$ operator self-adjoint? The results below give us answer the question.

Lemma 3.2: *Let* $T \in B(H)$ *be* $A - self\ adjoint$ *operator. Then T is self-adjoint if and only if T and* T^* *commute with an involution.*

Proof: Suppose $T\ is\ A - self\ adjoint$. Then $T^* = ATA^{-1}$ for some invertible and self-adjoint operator A. Now suppose that the similarity transformation A is an involution. Then, clearly, $ATA^{-1}T^* = A^{-1}T^*A = AT^*A^{-1}$. This assertion proves that $T = T^*$ and so T is self-adjoint.

Theorem 3.3 [15]: *Let H and K be Hilbert spaces and let* $A \in B(H, K)$. *Then*

i. $Ker(A) = Ran(A^*)^{\perp}$

ii. $Ker(A^*) = Ran(A)^{\perp}$

iii. $\overline{Ran(A)} = Ker(A^*)^{\perp}$

iv. $\overline{Ran(A^*)^{\perp}} = Ker(A)^{\perp}$

Remark: We note that if $A \in B(H)$ is self-adjoint, then by iii above, $\overline{Ran(A)} = Ker(A)^{\perp}$ and so $H = Ker(A) \oplus \overline{Ran(A)}$.

It has been proved in [7] that if $T \in B(H)$ is an $A - self\ adjoint$, then its adjoint T^* is injective. This result together with the corollary to Theorem 4.12 [13] enables us identify the relationship between $A - self\ adjoint$ operators and the quasi-affinity. (See Theorem 3.5 pp. 10, of [7]).

Evidently, if $T \in B(H)$ is an $A - self\ adjoint$ operator, then T and its adjoint, T^* are quasi-affinities. In fact T and T^* are left invertible, that is if there exists an operator$S \in B(H)$ such that $ST = I$ and $ST^* = I$.

We shall also give the relationship between metrically equivalent operators and A –unitarily equivalent operators for some given quasiaffinity:

Theorem 3.4: [10, Theorem 3.29 (ii)]: *If A and B are metrically equivalent operators and A is self-adjoint, then* $A = |B|$.

Theorem 3.5 [9, Theorem 2.9 (Fuglede-Putnam-Rosenblum)]: *Let* $A \in B(H)$ *and* $B \in B(H)$ *.If* $AX = XB$ *holds for some operator X, then* $A^*X = XB^*$.

Theorem 3.6: *Let* $A, B \in B(H)$. *Suppose A and B are metrically equivalent operators,* $AA^* = BB^*$ *and* $XB = AX, X^*B = AX^*$ *for some quasiaffinity X which is A-unitary, then A and B are A –unitarily equivalent.*

Proof: We first note that every unitary operator is $A - unitary$. We show that if A and B are metrically equivalent then they are unitarily equivalent.

Suppose $A^*A = B^*B$, $AA^* = BB^*$ and $XB = AX$, $X^*B = AX^*$ for some quasiaffinity X. Suppose $X = U|X|$ is the polar decomposition of X, where U is a partial isometry and $|X| = \sqrt{X^*X}$ is positive.

Define $W = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ and $S = \begin{pmatrix} 0 & A \\ B^* & 0 \end{pmatrix}$ on $H \oplus H$. Since X is a quasiaffinity, so is W. Using $XB = AX\ and\ X^*B = AX^*$ we have that $S^*S = \begin{pmatrix} B^*B & 0 \\ 0 & A^*A \end{pmatrix} = \begin{pmatrix} A^*A & 0 \\ 0 & B^*B \end{pmatrix} = SS^*$ and $SW = WS^*$ which means that S and S^* are quasisimilar normal operators. By the Fuglede-Putnam-Rosenblum Theorem above, S and S^* are unitarily equivalent meaning that there exists a unitary operator U such that $S = U^*S^*U$ where U is a polar decomposition of X. That is $\begin{pmatrix} 0 & A \\ B^* & 0 \end{pmatrix} = U^*\begin{pmatrix} 0 & B \\ A^* & 0 \end{pmatrix}U$, which shows that $A = U^*BU$.

Question: Is every part of an $A - self\ adjoint$ operator T also A-self adjoint? This question can be answered if we decompose T as a direct sum $T = T_1 \oplus T_2$ by specifying certain conditions on the direct summands of T .We summarize this in the following theorem:

Theorem 3.7: *Every part of an* $A - self\ adjoint$ *operator T is A-self adjoin*t.

Proof: Suppose $T = T_1 \oplus T_2$ where T_1 has a certain property P while T_1 is devoid of property P . Then by definition of A-self adjointness we have $T^* = T_1^* \oplus T_2^* = A(T_1 \oplus T_2)A^{-1} = AT_1A^{-1} \oplus AT_2A^{-1}$. Thus, $T_1 = AT_1A^{-1}$ and $T_2 = AT_2A^{-1}$ as required.

Remark: It has been shown in [7] that if $T \in B(H)$ is an $A - self\ adjoint$ operator then T is unitary if T is an involution. In additional, the spectrum of T is either real or complex; if complex, then the eigen values come in complex conjugate pairs.(see [6]). This gives us a necessary and sufficient condition for A-self adjointness.

In general, such operators have are symmetric with respect to the real axis. Equality of spectra is a necessary condition for A-self adjointness. We summarize it in the following corollary:

Corollary 3.8: *Let* $T \in B(H)$ *is an* $A - self\ adjoint$. *Then*

a). $\sigma_p(T) = \sigma_p(T^*)$

b). $\sigma_c(T) = \sigma_c(T^*)$

c). $\sigma_r(T) = \sigma_r(T^*)$

Proof: Since T is an $A - self\ adjoint$ then by definition $T^* = ATA^{-1}$. Thus, T^* and T are similar and hence have the same spectrum. Therefore the above claims follow since $\sigma(T)$ is the disjoint union of $\sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$.

Counter Example

The backward shift operator $T: l^2 \to l^2$ defined by $T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$ is not A-self adjoint. Its adjoint (called the unilateral shift) is defined by $T^*(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$ .We see (as an infinite matrix) that every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ (open unit disc centred at the origin) is in $\sigma_p(T)$ and that $\sigma_p(T^*) = \emptyset$. Also,

$\{ \lambda \in \mathbb{C}: |\lambda| < 1\} \subset \sigma_r(T^*)$. Hence T is not A-self adjoint (for any A with the required properties) because the necessary condition for A-self adjointness is not satisfied i.e. $\sigma(T) \neq \sigma(T^*)$.

Question: Given that $T \in B(H)$ is A-self adjoint, is AT −self adjoint? We provide the solution in the following theorem.

Theorem 3.9: *$T \in B(H)$ is A-self adjoint, if and only if is $AT -$ self adjoint.*

Proof: $T \in B(H)$ A-self adjoint implies that $T^* = ATA^{-1}$. We then have that $T^*A = AT$. Thus $(AT)^* = T^* A^* = T^*A = AT$ (since A is self-adjoint).
Conversely, let AT be A-self adjoint. Then $(AT)^* = AT$. Post multiplying both sides of this equation by A^{-1} and using the definition we have $T^* = ATA^{-1}$. This completes the proof.

Remark: In view of the above theorem, we see that the mapping defined by $\varphi: T \to AT$ is an isomorphism i.e. it establishes a one-to-one correspondence between the class of self- adjoint and A-self adjoint operators in the Hilbert space H. In fact if we let $T \in B(H)$ to be A-self adjoint then we see that T is self-adjoint if A commutes with T i.e. $AT = TA$. Here $T^* = ATA^{-1} = T$. Then $AT = TA$.

4. *A*-Self-Adjoint, Unitary Equivalence and *A*-Unitarily Equivalence of Operators

It is well known that unitary equivalence is an equivalence relation. We give a condition which shows that unitary equivalence preserves A-self adjointness.

Theorem 4.1: *Let S and T be bounded linear operators on a Hilbert space H. If T is A-self adjoint and T is unitarily equivalent to S, that is $UT = SU$, where U is a unitary operator, then S is UAU^*-self adjoint.*

Proof: We have $T^* = ATA^{-1}$ and $T = U^*SU$ for some unitary operator U. Using these two equations we can simplify and re-write S^* in terms of operators U, S and U^* only as:
$S^* = UT^*U^* = U[ATA^{-1}]U^* = U\{A(U^*SU)A^{-1}\}U^* = (UAU^*)S(UA^{-1}U^*)$ which establishes the claim.

Remark: The above theorem shows that unitary equivalence preserves A-self adjointness if and only if $UAU^* = A$. That is, if the unitary operator U is A^*-unitary.
We see that unlike self-adjointness, unitary equivalence does not preserve A-self adjointness.
The following results will enable us establish the relationship be A-unitarily equivalence and A-normal operators.

Definition 4.2: *The automorphism group of A-unitary operators is the set* $\mathbb{G}_A = \{T \in B(H): T^{[*]} = T^{-1}\}$.

Theorem 4.3 [7]. *Every unitary operator is A-unitary.*
Proof: [7, pp. 21].

Remark: $\mathbb{G}_A$ is a multiplicative group. If, $S \in \mathbb{G}_A$, then $ST \in \mathbb{G}_A$. This follows from $(ST)^{[*]} = T^{[*]}S^{[*]} = A^{-1}T^*S^*A = A^{-1}(AT^{-1}A^{-1}.AS^{-1}A^{-1})A = T^{-1}S^{-1} = (ST)^{-1}$.

Definition 4.4: Two linear operators $T \in B(H)$ and $S \in B(K)$ are said to be $A - unitarily\ equivalent$ (denoted $T \cong S$), if there exists an $A -$ unitary operator $U \in \mathcal{G}(H, K)$ such that $TU = US$.

In a real Hilbert space of dimension n, an operator is called Lorentz if it is $A - unitary$ where $A = I_p \oplus -I_q$ where $p, q \in \mathbf{N}$ and $p + q = n$. For instance if $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ is Lorentz.

Definition 4.5: A conjugation is a conjugate-linear operator $C: H \to H$ which is both involutory (i.e., $C^2 = I$) and isometric.

Remark: If we let $A = A^* = A^{-1}$, then A is a conjugation. Thus, this Jordan algebra $\mathbb{J}_A$ will contain the invertible normal operators, operators defined by Hankel matrices, Toeplitz and the Volterra integration operator $V(f)(t) = \int_0^t f(s)ds$ for a function $f(s) \in L^2(0,1)$ and $t \in (0,1)$.

Remark: Every A-unitary operator T is invertible. We note that if T is A-unitary then T^* is also A-unitary. This follows from the fact that $(T^{[*]})^* = (T^{-1})^* = (T^*)^{[*]} \Rightarrow (T^*)^{[*]} = (T^*)^{-1} \Rightarrow T^*$ is A-unitary.

Theorem 4.6 [8]: *If T is a normal operator and $S \in B(H)$ is unitarily equivalent to T, then S is normal.*

Proof: [8].

Theorem 4.7 [7]: *Every normal operator T is A-normal.*

Proof: [7, pp. 30-31].

Remark: Not all A-normal operators are normal. For example, if $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$and $T = \begin{pmatrix} i & i \\ i & 0 \end{pmatrix}$ a quick mathematical computation reveals that $T^{[*]}T = TT^{[*]}$ and $T^*T \neq TT^*$. Therefore, T is A-normal but not normal.

We also see that A-self adjoint and A-unitary operators are special cases of A-normal operators.

Corollary 4.8: *If T is an A - normal operator and $S \in B(H)$ is A- unitarily equivalent to T then S is A-normal.*

Proof: From Theorem 4.7 above, every unitary operator (w.l.o.g, letting $A = I$) is A- unitary and using a similar argument, we see that every normal operator T is A-normal. It suffices to show that S is normal.

Now, suppose that $SX = XT$, that is $S = XTX^*$ where X is $A -$ unitary and T is A-normal.

Then $S^*S = (XTX^*)^*(XTX^*) = XT^*X^*XTX^* = XT^*TX^*$ (Since $X^*X = I$) $= XTT^*X^*$ (Since T is normal) $= XT(XT)^* = SXX^*S^*$ (Since XT=SX and $(XT)^* = X^*S^*$) $= SS^*$ (Since $X^*X = I$). That is S is normal. Since every normal operator S is A-normal, it follows that S is A-normal as required.

Finally, we discuss some conditions that guarantee a product of A-self adjoint operators to be A-self adjoint:

Theorem 4.9: [7, Theorem 3.19 (ii)] *If P and Q are A-self adjoint operators, then the product $T = PQ$ is A-self adjoint if and only if $[P, Q] = 0$.*
By the above Theorem, we note that $\mathbb{J}_A$ is a linear space which is not closed under multiplication. However, it is closed with respect to the Jordan product given by the

equation $\{P,Q\} = \frac{1}{2}\{PQ + QP\}$.

Corollary 4.10: *An invertible operator T is a product of A-self adjoint operators P and Q if and only if T is A-self adjoint.*

Proof: Suppose T is invertible with $T = PQ$ and $P^* = APA^{-1}$,$Q^* = AQA^{-1}$. Invertibility of T implies that $I = TT^{-1} = (PQ)(PQ)^{-1} = PQQ^{-1}P^{-1}$ and $0 \notin \sigma(T)$ implies that $0 \notin \sigma(PQ)$. Hence P and Q are invertible and so is QP. Clearly, $T^* = (PQ)^* = Q^*P^* = (AQA^{-1})(APA^{-1}) = A(QP)A^{-1}) = A(PQA^{-1}$(Since$[P,Q] = 0$).That is $T^* = A(PQ)A^{-1} = ATA^{-1}$ which shows that T is $A - self\ adjoint.$

Conversely, suppose T is invertible and T is A-self adjoint. Since T is invertible, by the polar decomposition theorem, T has a unique polar decomposition $T = UM$, where U is unitary (and not necessarily self-adjoint) and $M = (T^*T)^{1/2}$ is positive (hence self-adjoint) operator. We use A -self adjointness of T to show that U, must indeed, be self-adjoint. A -self adjoint of T implies that $UM = A^{-1}(UM)^*A = A^{-1}MU^*A$, for some invertible operator $A.$ A-self adjoint of $T = UM$ (invertible) implies that U is self adjoint. But every self adjoint operator is A-self adjoint. This completes the proof.

Potential Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Acknowledgements

The author wishes to express his heartfelt gratitude to the referees for valuable comments and suggestions during the writing of this manuscript.

References

[1] Cassier G, Mahzouli H. and Zeroiali E. H, Generalized Toeplitz operators and cyclic operators, Oper. Theor. Advances and Applications 153 (2004): 03-122.

[2] Kubrusly C. S., An Introduction to Models and Decompositions in Operator Theory. Birkhäusers, Boston, 1997.

[3] Kubrusly C. S., Hilbert Space Operators, Birkhäusers, Basel, Boston, 2003.

[4] Lins B, Meade P, Mehl C and Rodman L. Normal Matrices and Polar decompositions in infinite Inner Products. Linear and Multilinear algebra, 49: 45-89, 2001.

[5] Mehl C. and Rodman L. Classes of Normal Matrices in infinite Inner Products. Linear algebra Appl, 336: 71-98, 2001.

[6] Mostafazadeh A., Pseudo-Hermiticity versus PT-symmetry, III, Equivalance of pseudo-Hermiticity and the presence of antilinear symmetries, J. Math. Phys. 43 (8) (2002), 3944-3951.

[7] Nzimbi B. M, Pokhariyal G. P and Moindi S. K, A note on A-self-adjoint and A-Skew adjoint Operators, Pioneer Journal of Mathematics and Mathematical sciences, (2013), 1-36.

[8] Nzimbi B. M, Pokhariyal G. P and Moindi S. K, A note on Metric Equivalence of Some Operators, Far East Journal of Mathematical sciences, Vol 75, No. 2 (2013), 301-318.

[9] Nzimbi B. M., Khalagai J. M. and Pokhariyal G. P., A note on similarity, almost similarity and equivalence of operators, Far East J. Math. Sci. (FMJS) 28 (2) (2008), 305-317.

[10] Nzimbi B. M, Luketero S. W, Sitati I. N, Musundi S. W and Mwenda E, On Almost Similarity and Metric Equivalence of Operators, Accepted to be published by Pioneer Journal of Mathematics and Mathematical sciences(June 14,2016).

[11] Patel S. M., A note on quasi-isometries II Glasnik Matematicki 38 (58) (2003), 111-120.

[12] Rehder Wulf, On the product of self-adjoint operators, Internat. J. Math. and Math. Sci 5 (4) (1982), 813-816.

[13] Rudin W, Functional Analysis, 2nd ed., International Series in Pure and Applied Math., Mc Graw-Hill's, Boston, 1991.

[14] Suciu L, Some invariant subspaces of A-contractions and applications, Extracta Mathematicae 21 (3) (2006), 221-247.

[15] Sz-Nagy B, Foias C, Bercovivi H and Kerchy L, Harmonic Analysis of Operators on Hilbert Space, Springer New York Dordrecht London (2010).

[16] Tucanak M and Weiss G, Observation and Control for Operator Semi groups Birkhauser, Verlag, Basel, 2009.

[17] Virtanen J. A: Operator Theory Fall 2007.

[18] Yeung Y. H, Li C. K and L. Rodman, on H-unitary and Block Toeplitz H-normal operators, H-unitary and Lorentz matrices: A review, Preprint.