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The Research of Discrete Mean - Variance Portfolio Problem with Time-Delay

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Abstract: Due to the financial sector complicated variety of events, each financial problems from changes to know its essence, the change rule, from the change of strategy to formulate relevant policy and policy into effect, etc., the process inevitably has a certain lag. Therefore, in order to better reflect the actual situation, we study the portfolio model with delays in this paper. By joining our delay control item, the optimization model was established, the goal is to maximize earnings expectations. In this paper, it studies the continuous time without delay the mean - variance portfolio problems on the basis of existing research. It established auxiliary problem using the stochastic linear quadratic optimal control theory. Using the maximum principle, the solution of the optimal investment strategy are given and it analysis the case, the conclusion is in conformity with the actual. It studies the existing time delay portfolio strategy problem in discrete time case. Based on the stochastic LQ (linear quadratic) optimal control theory, it established the discrete time model with time delay. The paper has carried on the solution and example analysis.

Keywords: The Mean - Variance Model, Portfolio Investment, Input Delay, Optimal Control, Investment Strategy

1. Introduction

In today's capital market, the pursuit of profit maximization is the main intrinsic motivation of each investor, but investment is risky, the returns and risk of venture capital is a problem of two aspects, along with investment of high yield and a corresponding must be high risk. The expected return and risk of portfolio depends on each of the securities income ratio, variance, the relative proportion of all kinds of securities and the correlation degree between the securities. Under certain conditions of the relevance of the various types of securities and securities gains of variance, investors can by adjusting the proportion of various securities purchase to reduce risk.

Due to the investment for risky assets, it needs to solve two major problems: expected return and risk, therefore, investors need to consider how to measure the risks and benefits of investment portfolio and how to balance these two indicators in the capital markets, and then invest in asset allocation. According to the two main problem, Markowitz in 1952 published paper in the Journal of Finance of Portfolio Selection [1], it systematically put forward the mean variance Portfolio theory in the first time, it provides a method to solve the problem of single stage investment Portfolio, the prototype of the mean - variance model. According to investment target, mean - variance model of single phase refers to investors at a particular moment to asset allocation, after that it keep to the end of the investment. Markowitz gives the numerical solution of mean-variance model in the market of short sales restrictions. Then many scholars has carried on the promotion and development on this basis. When markets allow short-selling, Merton [2] and $Szeg\ddot{o}$ [3] are given the model of single phase that does not contain the risk-free asset and contain the risk-free asset, in that situation it analysis the solution of the optimal portfolios and efficient frontiers. Among them, it mainly use Lagrange multiplier method to do that in the situation of does not contain risk-free assets. In the case of risk-free assets, it mainly use iterative method and genetic algorithm to do that, etc. See [4]. In the case of not allow short-selling, it can be solved by numerical method. Such as, Wen-jing Guo [5] gives a method of neural network algorithm, De-quan Yang [6] gives a range search method to solve the efficient portfolio frontier, etc. Zhu-wu wu [7] analysis the effective frontier drift in the situation of joining effective securities or eliminating invalid securities cases. He established the mean - variance model, using variance to measure the investment risk. The advantage is that the model is easy to understand, the relationship between the returns and risk can be directly said, but the variance deal with excess returns that higher than the average as risk, obviously it is not reasonable. Therefore, many scholars consider the second half of the variance is used to measure risk, Markowitz [8] study the average - the second half of the variance model. On average - the second half of the variance combination optimization problem, O. L. V. Costa presents a linear matrix Inequality method, Enrique Ballestero [10] study the effective frontier of the average - the second half of the variance problem.

As capital markets increasingly complex, single stage portfolio model can't meet the actual needs, in order to obtain high yield, investors tend to adjust investment strategy according to the change of market environment, the multi-stage investment, the resulting the multi-stage mean variance portfolio model. F. C. Jen, S. Zionts study found that compared with the single phase model, objective function in the multi-phase planning model contains Integral item $(E X_T)^2$ which has dynamic planning sense, make model is unable to get analytic solution. The following nearly 30 years, dynamic variance model studies were not made greater progress. Until 1998, D. L I, l. Wan [12] used maximize the expected utility function method to solve a mean - variance model, That is maximum *E[U(X(T))]*, *U* is the utility function, can be a form of quadratic form, the logarithmic form or index, etc., thus embedded in the question to another problem that can be solved by dynamic programming method, this is multi-phase mean - variance problem study of major breakthrough. In addition, Yao Haixiang [13] study the of multi-stage mean variance portfolio selection problem in the risky asset return on related cases. M. V. A raujo [14] study the parameter multi-phase mean - variance model in the market in line with the random market state transition process cases, get effective strategy is a closed set of conclusions and the optimal strategy by solving a set of coupled Riccati difference equation.

If market transactions is continuous, investors can adjust investment strategy at any point in time, it is called a continuous time mean - variance model. That is a multi-stage model for further extension. G. Yin, X. Zhou [15] reveals the discrete time me an - variance model and the nature of continuous time variance model, makes the study of multiphase portfolio naturally transition to the continuous time model. Capital market is full of randomness and uncertainty, how to seek the optimal strategy in the random environment become a research focus in the academic circles. X. Zhou [16] in a random linear quadratic optimal control theory for the tool, through the establishment of auxiliary problem, studies the continuous variance optimization problem under the complete market, This is the first time to embed the variance problem solving stochastic LQ optimal control problem, for subsequent use of the optimal control problem to solve the problem of portfolio laid a foundation. X. Zhou [17] gives the exact form of continuous time mean - variance model efficient frontiers; A. E. B. Lim, X. Zhou [18] and W. Guo [19] studies the investment decision making problems in the random market parameters, respectively, considering the stock prices have two kinds of situations of jumped and continuous, also using the stochastic LQ optimal control and backward stochastic differential equation method, getting the analytical form of effective investment strategies and efficient frontiers. X. Li, X. Zhou, A. E. B. Lim [20] studies the mean - variance optimal combination problem In case of not allow short-selling; for the continuous-time portfolio selection optimization problem in the case of liabilities, C. Mei, D. Li [21], S. Xie [22] gives the corresponding research results; L. Liu [23] considering the optimal portfolio selection problem in case of liabilities and not allow short-selling restrictions. In addition, R. Bielecki [24] respectively studied continuous time mean - variance optimal combination problem in case of not allowed to fail on the complete market and the market under the state transition; Since the mean-variance model only considering the investors' investment behavior, without considering the consumption behavior of investors and the influence of consumption to investment, it's inspired researchers through the utility function combined investors' investment behavior and consumer behavior. The purpose of the investors is pursuit of consumer utility and eventually the expected utility of wealth is the largest. Merton [25] [26] studied the optimal consumption investment strategy in the case of continuous time, investment spending under stochastic interest rate environment, investment spending under stochastic volatility model systematically, such as specific see S. H. Wang [27], etc.

With the deepening of the study, use the brown movement is not well characterized increasingly complex capital market, many scholars studied the underlying assets to case of diffusion process, jump diffusion process portfolio problem, specific see J. C. Cox [28], O. S. Alp [29], D. M. Dang [30], Y. L. Although Markowitz's mean variance model has important historical significance, but the assumption is too idealistic to market conditions. Therefore, scholars by various ways to relax assumption of the model, search for more close to the actual market conditions the optimal portfolio; Especially in recent years, the research of scholars closer to the actual situation of capital market, such as, Y. Xu, Z. Wu [31] is studied under the imperfect market have inflation mean variance portfolio problems; W. Pang [32] part studied the variance problem under the circumstance of information; S. Yang [33] studied under the complete market circumstances unbounded mean variance portfolio problem with random parameters.

At present, the research of time-delay stochastic LQ (linear quadratic) optimal control problem has made great progress and gets some effective method. For example, L. Chen, Z. Wu [34] gave maximum principle and its application of time-delay stochastic optimal control problem. X. Song, H. Zhang, L. Xie [35] studied for a class of discrete-time stochastic systems with input delay and meet the conditions of the optimal controller is given, and got the result of the linear quadratic control. Juanjuan Xu [36] studied continuous-time linear systems with input time-delay optimal control problem systematically. She get the necessary and sufficient condition for existence and uniqueness of the optimal solution and the display solution of the optimal controller based on the stochastic maximum principle. H. Zhang, L. Li [37] studied LQ optimal control problem with multiplicative noise and Input delay discrete time stochastic systems and get the display solution of the optimal controller based on the discrete stochastic maximum principle; More research about stochastic control are S. Chen, X. Li, X. Zhou [38], C. Li [39], etc.

Due to the Complex variability of financial sector and financial event, the releasing and spread of policies and message need a certain amount of time, which makes a time lag when investors adjust the investment strategy. In order to better reflect the actual situation, in this paper, we study portfolio model with time delay in order to realize the expected utility maximization goals. In theory, research mean - variance portfolio problem with time-delay with the help of LQ optimal control method with input delay in control theory. On the one hand, adding time delay widen the depth of the mean - variance portfolio problem, which makes the model more close to the actual situation and improves the theoretical model applied in the real problems. On the other hand, using the control theory research results to solve the problem of financial theory, which makes control theory having a new place and also inspires further research. On the basis of existing research, this paper presents continuous time mean-variance model without time delay, through establishing of auxiliary problem, the original problem is transformed into LQ control problem, and then use stochastic LQ optimal control method to get the analytic solution of the optimal portfolio strategy. According to multi-phase mean-variance portfolio model, established discrete time variance model with time-delay, gave the solution of general form discrete LQ optimal control problem with time delay, analyzed of the example.

2. Prepare Knowledge

The mean-variance portfolio problem

In 1952[1], the famous American scholar Markowitz established the mean-variance model according to the relationship between the yield of risky assets and risk firstly. This model mainly solves the investors how to select an optimal portfolio problem from all possible portfolio in the beginning. Markowitz believe that investors' decision goal should be two, that is as high as possible returns and as low as possible risks, the best decision-making should be the two goals to achieve the best balance.

Definition 2.1 If a portfolio has maximum expected yields to determine the level of variance, at the same time, has the smallest variance to determine the expected yield level, this portfolio is called the mean variance portfolio effectively.

2.1. Single Phase Variance Model

Assuming that financial markets has n kind of assets that available for trading. Yields are respectively $X_1, X_2, ..., X_n$, $\mu = E(X) = (EX_1, EX_2, ..., EX_n)^T$ is mathematical expectation vector of earnings. The covariance of assets yield is $\sigma_{ij} = \operatorname{cov}(X_i, X_j)$ $(i, j = 1, ..., n)$, X corresponds to the covariance matrix is $\Sigma = (\sigma_{ij})_{n\times n}$, here assumes that Σ is a degradation of matrix. In particular, the vector $1 = (1,1,...,1)^T$. If investors have the capital of one unit, he invested in the portfolio vector n kind of risky assets is $\Pi = (\pi_1, \pi_2, ..., \pi_n)^T$, $\sum_{i=1}^{n} \pi_i = 1$, π_i is the proportion of investment in the i th kind of assets. The total yield of portfolio is $X_P = \sum_{i=1}^{n} \pi_i X_i$. The total yield of volatility as a risk, then $\operatorname{var}(X_p) = \sigma^2 = \Pi^T \Sigma \Pi$. Suppose the market utility function of investors is the mean square of the utility function, when investors choose portfolio, if the expected to yield r_p is certain, select portfolio model that makes its minimum risk is as follows:

$$\min \frac{1}{2}\sigma_p^2 = \frac{1}{2}\Pi^T \Sigma \Pi$$

$$\text{S.T} \begin{cases} E(X_p) = \mu^T \Pi \geq r_p \\ \sum_{i=1}^{n} \pi_i = \Pi^T 1 = 1 \end{cases}$$

Lagrange multiplier method was applied to solve the above problem, then

$$L = \frac{1}{2}\Pi^T \Sigma \Pi + \lambda_1 (r_p - \Pi^T \mu) + \lambda_2 (1 - \sum_{i=1}^{n} \pi_i)$$

λ_1, λ_2 of them are undetermined parameters, the optimal solution of the first-order condition is:

$$\frac{\partial L}{\partial \Pi^T} = 0, \frac{\partial L}{\partial \lambda_1} = 0, \frac{\partial L}{\partial \lambda_2} = 0 \tag{1}$$

Then we can get the optimal solution is:

$$\Pi^* = \Sigma^{-1}(\lambda_1 \mu + \lambda_2 1) \tag{2}$$

So, it's available that:

$$r_p = \lambda_1 \mu^T \Sigma^{-1} \mu + \lambda_2 \mu^T \Sigma^{-1} 1 = \lambda_1 a + \lambda_2 b \tag{3}$$

$$1 = \lambda_1 1^T \Sigma^{-1} \mu + \lambda_2 1^T \Sigma^{-1} 1 = \lambda_1 b + \lambda_2 c \tag{4}$$

Among them $a = \mu^T \Sigma^{-1} \mu$, $b = \mu^T \Sigma^{-1} 1 = 1^T \Sigma^{-1} \mu$ $c = 1^T \Sigma^{-1} 1$, let $\Delta = ac - b^2$

The solution of equations (3) (4): $\lambda_1 = \frac{r_p c - b}{\Delta}, \lambda_2 = \frac{a - r_p b}{\Delta}$

Put λ_1, λ_2 in (2), the variance of the optimal portfolio are as follows:

$$\sigma_p^2 = \Pi^{*T}\Sigma\Pi^* = \frac{cr_p^2 - 2br_p + a}{\Delta} = \frac{c}{\Delta}(r_p - \frac{b}{c})^2 + \frac{1}{c} \quad (5)$$

We can see that the equation is a parabolic equation by equation (5). Components which included in parabolic make up the feasible set of portfolio, the upper branch of the parabola is called the efficient frontier. In all possible scenarios of the portfolio provided by feasible set, investors can find effective set through effective set theorem.

2.2. Multi-stage Model

Multi-stage model is an extension of the model of single phase. For investors, the investment behavior tends to be long-term. Investors will adjust the investment strategy along with the change of market environment, so that under the condition of a given final earnings makes the minimum variance or under a given variance levels causes the final yield is the largest. Investors' investment strategy is portfolio group constituted by each stage portfolio that generally can be solved by using the dynamic programming principle.

Let $r_t^i (i = 0,1,...,n)$ is the i th kind of asset's random rate of return in the t phase, $\pi_t^i (i = 0,1,...,n)$ denote the proportion of the i th kind of assets invested in the t phase, X_t denote the amount of wealth at the end of the t investment stage and investors are in T phase of the investment. Let $r_t = (r_t^1, r_t^2, ..., r_t^n)^T, \Pi_t = (\pi_t^1, \pi_t^2, ..., \pi_t^n)^T, t = 1,2,...,T$. Then the multi-stage model is:

$$\min_{\Pi} Var(X_T)$$

$$\text{S.t} \begin{cases} E(X_T) \geq \mu \\ X_t = X_{t-1}[1 + r_t^T \Pi_t + (1 - \Pi_t^T 1_n) r_t^0] \\ t = 1,2,...,T \end{cases} \quad (6)$$

Among them μ is the final expected yields.

Although the multi-stage portfolio model have already a long time, it is difficult to get the analytical solution because its objective function contains $(EX_T)^2$ integral item of dynamic planning sense, therefore we need to introduce approximation problem to solve this difficult problem.

2.3. Continuous Time Model

Continuous time model is further extension of multi-stage model. It assumes that market change over time continuously. Suppose that there are $m+1$ kinds of assets trading continuously and one of these is a risk-free securities, its price $P_0(t)$ change process to satisfy the following differential equation:

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt, t \in [0,T] \\ P_0(0) = p_0 \succ 0 \end{cases} \quad (7)$$

Among them $r(t) \succ 0$ is risk-free securities rates, and the rest m kinds are risk securities, the price process to satisfy the following stochastic differential equation:

$$\begin{cases} dP_i(t) = P_i(t)[b_i(t)dt + \sum_{i=1}^{m} \sigma_{ij}(t)dw^j(t)], t \in [0,T] \\ P_i(0) = p_i \succ 0 \end{cases} \quad (8)$$

Among them $\sigma_i(t) = (\sigma_{i1}(t), \sigma_{i2}(t), ..., \sigma_{im}(t))$ is the volatility of risk assets i, $b_i(t) \succ 0$ is expected yields of the i th kind of asset and $\sigma_t = (\sigma_{ij}(t))_{m\times m}, b(t) = (b_1(t), b_2(t), ..., b_m(t))$. Assume that $r(t) \in C([0,T]; R), b(t) \in C([0,T]; R^m), \sigma_{ij}(t)_{m\times m} \in C([0,T]; R^{m\times m})$ are a F_t-measurable function; Assume that there exist constant $\delta \succ 0$ that makes $\sigma(t)\sigma(t)^T \geq \delta I, \forall t \in [0,T]$; Assume that all the functions on t are measurable and uniformly bounded.

Assume that wealth of investors at time of t is $x(t)$, the number of the i th kind of assets is $N_i(t)$, then

$$x(t) = \sum_{i=0}^{m} N_i(t)P_i(t), t \geq 0 \quad (9)$$

Assume that trading is continuous and do not consider the transaction cost and consumption, then

$$\begin{cases} dx(t) = \sum_{i=o}^{m} N_i(t)dP_i(t) \\ = \left\{ r(t)N_0(t)P_0(t) + \sum_{i=1}^{m} b_i(t)N_i(t)P_i(t) \right\} dt + \sum_{i=1}^{m} N_i(t)P_i(t) \sum_{j=1}^{m} \sigma_{ij}(t)dw_j(t) \\ = \left\{ r(t)x(t) + \sum_{i=1}^{m} [b_i(t) - r(t)]u_i(t) \right\} dt + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t)u_i(t)dw_j(t) \\ x(0) = x_0 \succ 0 \end{cases} \quad (10)$$

Among them $\mu_i(t) = N_i(t)P_i(t)(i = 0,1,...,m)$ is the investment share of the i th kind of assets in the time t, say $u(t) = (u_1(t), u_2(t),...,u_m(t))^T$ is a portfolio strategy of investors. Our goal is to select the optimal investment strategy $\mu^*(t)$ makes a final moment of expected revenue $Ex(T)$ is the largest, and the corresponding risk $Varx(T) = E[x(T) - Ex(T)]^2 = Ex(T)^2 - [Ex(T)]^2$ is minimal.

3. Excluding Continuous Time Delay of the Mean - Variance Portfolio Investment Decisions

Then consider from the study of literature [16], under the framework of stochastic linear quadratic optimal control, consider excluding continuous time delay of the mean - variance portfolio strategy. Investor's goal is to pursue maximum return and minimum variance of expectation wealth at end moment. It can be turned into a single-objective stochastic control problem by giving two objectives empowerment and then converse into a standard linear quadratic problem.

3.1. Model Description and the Approximate Problem Building

Consider the continuous time mean - variance model given earlier, symbols used in front of said. In the continuous trading and do not consider the transaction cost and consumption cases, the wealth $x(t)$ of the investors in t time meet that

$$\begin{cases} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^{m} [b_i(t) - r(t)]u_i(t) \right\} dt + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t)u_i(t)dw^j(t) \\ x(0) = x_0 \succ 0 \end{cases} \quad (11)$$

Our goal is to choose an optimal investment strategy $u^*(t)$, makes the expected revenue $Ex(T)$ at final moments is the largest, at the same time corresponding to the minimum risk $Varx(T) = Ex(T)^2 - [Ex(T)]^2$, this is a double objective optimization problem. According to multi-objective optimization theory, it can be transformed double objective optimization problem into single objective optimization problem as follows:

$$\begin{cases} \min -Ex(T) + \mu Varx(T) \\ E\int_0^T u(t)^T u(t)dt \prec \infty \\ (x(t), \mu(t)) meet (11) \text{ type.} \end{cases} \quad (12)$$

The parameter $\mu \succ 0$ express risk factor, the above problem notes $P(\mu)$.

Due to the objective function of $P(\mu)$ contains $[EX(T)]^2$. It is inseparable under the dynamic planning and difficult to solve, therefore, so it introduce the following auxiliary problems:

$$\begin{cases} \min J(u(.);\mu,\lambda) = E\left\{\mu x(T)^2 - \lambda x(T)\right\} \\ E\int_0^T u(t)^T u(t)dt \prec \infty \\ (x(t), \mu(t)) meet (3.1) \text{ type.} \end{cases} \quad (13)$$

The parameter is $\mu \succ 0$, $-\infty \prec \lambda \prec +\infty$, the problem is $A(\mu,\lambda)$.

By literature [16], the solution set of problem $P(\mu)$ is subset of the solution set of problems $A(\mu,\lambda)$, so it can be solved by solving problem $A(\mu,\lambda)$ to solve the problem $P(\mu)$.

Converse the auxiliary problem (13) as following: $\gamma = \frac{\lambda}{2\mu}$ $y(t) = x(t) - \gamma$

The problem (13) is equivalent to the following questions:

$$\begin{cases} \min E[\frac{1}{2}\mu y(T)^2] \\ dy(t) = [r(t)y(t) + (b(t) - r(t))u(t) + \gamma r(t)]dt + \sum_{j=1}^{m}\sum_{i=1}^{m} \sigma_{ij}(t)u_i(t)dw^j(t) \\ y(0) = x_0 - \gamma \end{cases} \quad (14)$$

3.2 Solution of the General Form Stochastic LQ Problem

LQ problem is a class of important optimal control problem and it has a good structure, its optimal solution is obtained by solving the Riccati equation. It can solve the mean - variance model by stochastic LQ problem due to LQ structure of the mean - variance portfolio problem.

General form of linear stochastic differential equation is as follows:

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t) + f(t)]dt + \sum_{j=1}^{m} D_j(t)u(t)dw^j(t) \\ x(0) = x_0 \in R^n \end{cases} \quad (15)$$

For $\forall\ u(t) \in L_F^2([0,T];R^m)$, the corresponding value function is:

$$J(u(.)) = E[\int_0^T \frac{1}{2}[x(t)^T Q(t)x(t) + u(t)^T R(t)u(t)]dt + \frac{1}{2}x(T)^T Hx(T) \quad (16)$$

Among them $Q \in R^{n\times n}, R \in R^{l\times l}, H \in R^{n\times n}$ is certain weighting coefficient matrix.

In the form of the following equation is stochastic Riccati equation:

$$\begin{cases} \dot{P}(t) = -P(t)A(t) - A(t)^T P(t) - Q(t) + P(t)B(t)K(t)^{-1}B(t)^T P(t) \\ P(T) = H \\ K(t) = R(t) + \sum_{j=1}^{m} D_j(t)^T P(t)D_j(t) \succ 0, \forall t \in [0,T] \end{cases} \quad (17)$$

$$\begin{cases} \dot{n}(t) = -A(t)^T n(t) + P(t)B(t)(R(t) + \sum_{j=1}^{m} D_j(t)^T P(t)D_j(t))^{-1} B(t)^T n(t) - P(t)f(t) \\ n(T) = 0 \end{cases} \quad (18)$$

Theorem 3.1: if the $P(t)$ and $n(t)$ are continuous in the $[0,T]$, the stochastic LQ problem (15) - (16) has the optimal feedback control as following:

$$u^*(t,x) = -(R(t) + \sum_{j=1}^{l} D_j(t)^T P(t)D_j(t))^{-1} B(t)^T (P(t)x + n(t)) \quad (19)$$

Proof: consider the stochastic linear system, consider a one-dimensional situation for brevity.

$$dx(t) = (A(t)x(t) + B(t)u(t) + f(t))dt + (C(t)x(t) + D(t)u(t) + \overline{f}(t))dw(t)$$

$$x(0) = x_0 \quad (20)$$

The value function is

$$J_T = \frac{1}{2}E[\int_0^T (x(t)^T Qx(t) + u(t)^T Ru(t))dt + x(T)^T P(T)x(T) \quad (21)$$

By the maximum principle, along with state $p(t)$ meet the backward stochastic differential equations as follows:

$$\begin{cases} dp(t) = -[A(t)^T p(t) + C(t)^T q(t) + Qx(t)]dt + q(t)dw(t) \\ p(T) = P(T)x(T) \end{cases} \quad (22)$$

The optimal control $u(t)$ to satisfy equilibrium conditions as:

$$0 = Ru(t) + E[B(t)^T p(t) + D(t)^T q(t) | F_t] \quad (23)$$

Assuming that the relationship of accompany state and system state is as follows:

$$p(t) = P(t)x(t) + n(t)$$

So

$$dp(t) = \dot{P}(t)x(t) + P(t)dx(t) + \dot{n}(t)dt$$
$$= [\dot{P}(t)x(t) + P(t)(A(t)x(t) + B(t)u(t) + f(t)) + \dot{n}(t)]dt \quad (24)$$
$$+ P(t)(C(t)x(t) + D(t)u(t) + \overline{f}(t))dw(t)$$

The corresponding coefficient $dw(t)$ is equal, it is concluded that:

$$q(t) = P(t)(C(t)x(t) + D(t)u(t) + \overline{f}(t))$$

Put $p(t), q(t)$ respectively in equilibrium conditions, because the $x(t)$ is for F_t - measurable, concluded that

$$0 = Ru(t) + E\{B(t)^T (P(t)x(t) + n(t))$$
$$+ D(t)^T P(t)[C(t)x(t) + D(t)u(t) + \overline{f}(t)] | F_t\} \quad (25)$$
$$= (R + D^T(t)P(t)u(t) + [B(t)^T P(t) +$$
$$D(t)^T P(t)C(t)]x(t) + B(t)^T n(t) + \overline{f}(t)$$

Then the optimal controller can be represented as:

$$u(t) = -(R + D(t)^T P(t)D(t))^{-1}[(B^T(t)P(t) + D(t)^T P(t)C(t))x(t) + B(t)^T n(t) + \overline{f}(t)] \quad (26)$$

Due to the (22) and (24) in the corresponding coefficient of dt items are equal, and $C = 0, \overline{f}(t) = 0$, then it can get the conclusion of theorem.

3.3 Solution of the Auxiliary Model

Problem (14) is the special case of (15) (16), and consider that

$$A(t) = r(t), B(t) = (b_1(t) - r(t), ..., b_m(t) - r(t)), \quad f(t) = \gamma r(t),$$
$$(Q(t), R(t)) = (0,0),$$

$D_j(t) = (\sigma_{1j}(t), ..., \sigma_{mj}(t))$, $H = \mu$. $x(t)$ is one-dimensional, order that

$$\rho_t = B(t)[\sum_{j=1}^{m} D(t)_j^T D(t)_j]^{-1} B(t)^T = B(t)[\sigma(t)\sigma(t)^T]^{-1} B(t)^T \quad (27)$$

By (17)

$$\begin{cases} \dot{P}(t) = (\rho_t - 2r(t))P(t) \\ P(T) = \mu \\ P(t)[\sigma_t \sigma_t^T] \succ 0, t \in [0,T] \end{cases} \quad (28)$$

It is easy to get the solutions of (28):

$$P(t) = \mu e^{-\int_t^T (\rho(s) - 2r(s))ds} \quad (29)$$

The equation (18) can be written as:

$$\begin{cases} n'(t) = (\rho(t) - r(t))n(t) - \gamma r(t)P(t) \\ n(t) = 0 \end{cases} \quad (30)$$

The (19) shows that the optimal investment strategy

$$u^*(t,y) = -[\sigma(t)\sigma'(t)]^{-1} B(t)'[y + \frac{n(t)}{P(t)}] \quad (31)$$

Let $h(t) = n(t) / P(t)$, it can get

$$\dot{h}(t) = \frac{\dot{P}(t)n(t) - \dot{n}(t)P(t)}{P(t)^2} = r(t)h(t) - \gamma r(t)$$

And $h(T) = 0$, so it can get that

$$h(t) = \gamma(1 - e^{-\int_t^T r(s)ds}) \quad (32)$$

Put (32) into (31), considering the substitution, it can get the optimal control strategy is:

$$u^*(t,x) = [\sigma(t)\sigma'(t)]^{-1} B'(t)(\gamma e^{-\int_t^T r(s)ds} - x) \quad (33)$$

Put (33) into (11), it is concluded that:

$$\begin{cases} dx(t) = [(r(t) - \rho(t)x(t) + \gamma e^{-\int_t^T r(s)ds}\rho(t)]dt \\ +B(t)(\sigma(t)\sigma'(t))^{-1}\sigma(t)(\gamma e^{-\int_t^T r(s)ds} - x)dw(t) \\ x(0) = x_0 \end{cases} \quad (34)$$

Get expected available from both sides, it can get that:

$$\begin{cases} dEx(t) = [(r(t) - \rho(t))Ex(t) + \gamma e^{-\int_t^T r(s)ds}\rho(t)]dt \\ Ex(0) = x_0 \end{cases} \quad (35)$$

Solve (35), it can get that:

$$Ex(T) = \alpha x_0 + \beta\gamma \quad (36)$$

Among them $\alpha = e^{\int_0^T (r(t) - \rho(t))dt}, \beta = 1 - e^{-\int_0^T \rho(t)dt}$.

3.4. The Example Analysis

Consider an investor, the initial wealth is one million yuan, there are two alternative investment object, the one a risk-free securities $r = 0.05$, the other one is stock return risk $b = 0.15$. Assuming that invests for a year, $T = 1$, analysis example in the following three kinds of circumstances.

1) When the standard deviation is $\sigma = 0.2$, investors expect the yield of 20%. That is $Ex(1) = 1.2$ million yuan. By (28), it can easy get that $\rho = ((b-r)/\sigma)^2 = 0.25$, using (29) it can get $\gamma = (1.2 - e^{-0.2}) / (1 - e^{-0.25}) = 1.7238$,

Thus, by (33), the proportion of invested in risky assets is:

$$u^*(t,x) = 2.5(1.7238e^{0.05(t-1)} - x)$$

$u^*(t,x)$ is a function about time and wealth, when $t = 1, x = 1.2$, can calculate that $u^*(1,1.2) = 1.3095$.

2) When standard deviation is $\sigma = 0.2$ and given value γ is $\gamma = 1.2, 1.6, 1.8$ respectively, corresponding to the expected wealth value at end stage is $E(x) = 1.0842, 1.1726, 1.21169$ respectively, the proportion $\bar{u}(0,x_0)$ of zero time investing in risky assets is $\bar{u}(0,x_0) = 0.3537, 1.3049, 1.7805$ respectively. Thus, it can found that when risk assets volatility is certain, the bigger of γ, the more wealth the investors expect to obtain, the greater the amount of initial investment in risk assets.

3) When the terminal expected wealth value is $Ex(1) = 1.2$ million and the variance $\sigma = 0.15, 0.2, 0.25$, the corresponding γ is $\gamma = 1.4656, 1.7238, 2.0568$, the proportion $\bar{u}(0,x_0)$ of zero time investing in risky assets is $\bar{u}(0,x_0) = 1.7515, 1.5993, 1.5304$. Thus it can be seen that the expectation of investment income wealth under certain conditions, the greater the volatility of risk assets, and its corresponding value r is larger, the smaller the initial investment in the proportion of risky assets.

4. Discrete Time Variance Model with Time Delay

Because the actual problem exist the influence of factors such as access to information lag and formulate policy lag etc, so input control existed delay problems inevitably. This section in the basis of the above, according to the result of literature [42], established the corresponding discrete model, worked out the analytic solution of the optimal investment decision, and gave out the analysis of examples.

4.1. Description of Model

Set r_k^0 as risk-free asset return, $r_k^i(i=1,2,...,n)$ for the I th risky assets in the kth phase of the random rate of return, u_{k-d}^i said investors in $k-d$ phase invest to the proportion of the ith asset, σ_{ij} said the covariance between the risk assets and the random rate of return. x_k For investors to invest in the k phase at the end of the amount of wealth, investors to invest N stage, the purpose of that is to make final moment of expected wealth value maximum and risk minimum. With time delay control model of the discrete form expression is as follows:

$$\begin{cases} \max\limits_{u} E(x_T)-\mu Var(x_T) \\ x_{k+1}=(1+r_k^0)x_k+(R_k^i+\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{m}\sigma_{ij}w_k)u_{k-d} \end{cases} \tag{37}$$

The parameter $\mu \succ 0, R_k^i=r_k^i-r_k^0$.

Introduce the problem (37) auxiliary function is as follows:

$$\begin{cases} \min J=E[\mu x(T)^2-\lambda x(T)] \\ x_{k+1}=(1+r_k^0)x_k+(r_k^i-r_k^0+\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{m}\sigma_{ij}w_k)u_{k-d} \end{cases}$$

$$\mu \succ 0,-\infty \prec \lambda \prec +\infty \tag{38}$$

The parameter $\mu \succ 0,-\infty \prec \lambda \prec +\infty$

Let $\gamma=\dfrac{\lambda}{2\mu}, y_k=x_k-\gamma$, Problem (38) is equivalent to the following models:

$$\begin{cases} \min J=E[\frac{1}{2}\mu y(T)^2] \\ y_{k+1}=(1+r_k^0)y_k+(R_k^i+\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{m}\sigma_{ij}w_k)u_{k-d}+\gamma r_k^0 \end{cases} \tag{39}$$

w_k is white noise.

4.2. Stochastic Maximal Principle of Discrete System Containing Time Delay

Assume that feasible control field is convex set, as the following general control input of the controlled system containing time-delay

$$x_{k+1}=(A+w_k\overline{A})x_k+(B+w_K\overline{B})u_{k-d}+(f+w_k\overline{f}) \tag{40}$$

x_k is state of the system, u_{k-d} is control input containing time-delay, w_k is White Gaussian noise, Constant coefficient matrix

$$A,\overline{A}\in R^{n\times n}; B,\overline{B}\in R^{n\times m}; f,\overline{f}\in R^n .$$

With the system (40) corresponding to the value function is:

$$J=E\left\{\sum_{k=0}^{N}x_k'Qx_k+\sum_{k=d}^{N}u_{k-d}'Ru_{k-d}+x_{N+1}'P_{N+1}x_{N+1}\right\} \tag{41}$$

$Q,P_{N+1}\in R^{n\times n}$ is the half positive definite symmetric matrix of the corresponding dimension, $R\in R^{m\times m}$ is given symmetric positive definite matrices.

Lemma 4.1 System (40) necessary conditions such that J_N is minimum is

$$\lambda_{k-1}=E[(A+w_k\overline{A})'\lambda_k|F_{k-1}]+Qx_k \tag{42}$$

$$0=E\left\{(B+w_k\overline{B})'\lambda_k|F_{k-d-1}\right\}+Ru_{k-d}, k=0,...,N \tag{43}$$

(42) is called the adjoint equation, λ_k is called the adjoint state, terminal condition is met $\lambda_N=P_{N+1}x_{N+1}$, and P_{N+1}

Given by the value function (41). System equation (40) and the adjoint equation (42) is make up Is backward difference equation systems, (43) is called equilibrium equation.

Proof: Fixed terminal time N, and u_k for F_{k-s-1} measurable, consider LQ control problem, take admissible control sets:

$$U=\left\{\mu_k \middle| E|\mu_k|^2\prec+\infty, \mu_k\ for\ F_{k-s-1}\ measurable\right\}$$

Allow the control field is convex set, take $u_k,\delta u_k\in U,\varepsilon\in(0,1)$, then $u_k+\delta u_k\in U$.

The J_N,x_k denote under u_k the value function and state trajectory respectively, $J_N^\varepsilon,x_k^\varepsilon$ denote the objective function of the controller under u_k^ε and state trajectory, denote by: $\delta J_N=J_N^\varepsilon-J_N,\delta x_k=x_k^\varepsilon-x_k$, then for δJ_N as Taylor expansion, can get:

$$\delta J_N=E\left\{2x_{N+1}'P_{N+1}\delta x_{N+1}+2\sum_{k=0}^{N}[x_k'Q\delta x_k+u_k'R\varepsilon\delta u_k]+O\left(\varepsilon^2\right)\right\} \tag{44}$$

$O(\varepsilon^2)$ is ε^2 with infinitesimals.

By the system equation (40) available:

$$\begin{aligned}\delta x_{k+1} &= (A+\bar{A}w_k)\delta x_k + (B+\bar{B}w_k)\varepsilon\delta u_k \\ &= F_x(k,0)\delta x_0 + \sum_{i=0}^{k} F_x(k,i+1)(B+\bar{B}w_i)\varepsilon\delta u_i\end{aligned} \tag{45}$$

Among them $F_x(k,i) = (A+\bar{A}w_k)(A+\bar{A}w_{k-1})\ldots(A+\bar{A}w_i)$, $F_x(k,k+1) = \mathrm{I}$

Bring type (45) into type (44) available:

$$\begin{aligned}\delta J_N = E\Big\{&2x'_{N+1}P_{N+1}[F_x(N,0)\delta x_0 + \sum_{i=N}^{k} F_x(N,i+1)(B+\bar{B}w_i)\varepsilon\delta u_i] \\ &+2\sum_{k=0}^{N} x'_k Q F_x(k-1,0)\delta x_0 + 2\sum_{k=0}^{N} u'_k R\varepsilon\delta u_k + O(\varepsilon^2) \\ &+2\sum_{k=0}^{N} x'_k Q \sum_{i=0}^{k-1} F_x(k-1,i+1)(B+\bar{B}w_i)\varepsilon\delta u_i\Big\}\end{aligned} \tag{46}$$

Due to the last item of (46) can be expressed as

$$\sum_{k=0}^{N} x'_k Q \sum_{i=0}^{k-1} F_x(k-1,i+1)(B+\bar{B}w_i)\varepsilon\delta u_i = \sum_{i=0}^{N-1}\sum_{k=i+1}^{N} x'_k Q F_x(k-1,i+1)(B+\bar{B}w_i)\varepsilon\delta u_i \tag{47}$$

Notice $\delta x_0 = 0$, use the type (47) to calculate δJ_N as follows:

$$\begin{aligned}\delta J_N = E\Big\{&2x'_{N+1}P_{N+1}\sum_{i=0}^{N} F_x(N,i+1)(B+\bar{B}w_i)\varepsilon\delta u_i \\ &2\sum_{i=0}^{N-1}\sum_{k=i+1}^{N} x'_k Q F_x(k-1,i+1)(B+\bar{B}w_i)\varepsilon\delta u_i + 2\sum_{k=0}^{N} u'_k R\varepsilon\delta u_k\Big\} + O(\varepsilon^2) \\ &= E\left\{G(N+1,N)\varepsilon\delta u_N + \sum_{l=0}^{N-1} G(l+1,N)\varepsilon\delta u_l\right\} + O(\varepsilon^2) \\ &= E\big\{E[G(N+1,N)|F_{N-1}]\varepsilon\delta u_N\big\} + E\left\{\sum_{l=0}^{N-1} E[G(l+1,N)|F_{l-1}]\varepsilon\delta u_l\right\} \\ &+E\big\{\{G(l+1,N) - E[G(N+1,N)|F_{N-1}]\}\varepsilon\delta u_N\big\} \\ &+\left\{\sum_{l=0}^{N-1}\{G(l+1,N) - E[G(l+1,N)|F_{l-1}]\}\varepsilon\delta u_l\right\} + O(\varepsilon^2) \\ &= E\big\{E[G(N+1,N)|F_{N-1}]\varepsilon\delta u_N\big\} + E\left\{\sum_{l=0}^{N-1} E[G(l+1,N)|F_{l-1}]\varepsilon\delta u_l\right\} + O(\varepsilon^2)\end{aligned} \tag{48}$$

$G(N+1,N), G(l+1,N)$ Respectively:

$$G(N+1,N) = 2x'_{N+1}P_{N+1}(B+\bar{B}w_N) + 2u'_N R \tag{49}$$

$$G(l+1,N) = [2u'_l R + 2x'_{N+1}P_{N+1}F_x(N,l+1) + \sum_{k=l+1}^{N} x'_k Q F_x(k-1,l+1)](B+\bar{B}w_l) \tag{50}$$

By (48) know necessary conditions such that J is the smallest is:

$$E[G(N+1,N)|F_{N-1}] = 0 \tag{51}$$

$$E[G(l+1,N)|F_{l-1}] = 0 \tag{52}$$

Below prove (49) - (50) and (51) - (52) is equivalent. Bring type (43) into type (42), and $k = N$, then can get:

$$E[(B+\bar{B}w_N)'P_{N+1}x_{N+1} + Ru_N|F_{N-1}] = 0 \tag{53}$$

To compare (53) and (51), it can know that (53) that is (51); In addition, by (42) it know:

$$\lambda_{k-1} = E[(A+w_k\bar{A})'\lambda_k|F_{k-1}]+Qx_k$$

$$= E[(A+w_k\bar{A})'(A+w_{k+1}\bar{A})'\lambda_{k+1}+(A+w_k\bar{A})'Qx_k|F_{k-1}]$$

$$= E[\sum_{j=k}^{N} F_x(j-1,k)Qx_j + F_x(N,k)P_{N+1}x_{N+1}] \quad (54)$$

Bring type (54) into (43) can get:

$$0 = E\{(B+\bar{B}w_k)'[Ru_k + \sum_{j=k}^{N} F_x(j-1,k)Qx_j$$

$$+F_x(N,k)P_{N+1}x_{N+1}+P_{N+1}x_{N+1}]|F_{N-1}\} \quad (55)$$

It's easy to see the (55) that is (52) q.e.d.

4.3. Solving of General Discrete LQ Problem Containing Time Delay

Representation of general form of containing time-delay discrete LQ problem is as follows:

$$x_{k+1} = (A+w_k\bar{A})x_k + (B+w_k\bar{B})u_{k-d} + (f+w_k\bar{f}) \quad (56)$$

w_k is one dimensional Gaussian white noise of zero mean and σ^2 variance, $x_k \in R^n$ is the system state, $u_k \in R^m$ is the control input, $d \succ 0$ is the time delay, A、B、$\bar{A}$、$\bar{B}$ is the corresponding dimension constant matrix; f、$\bar{f}$ be given certain function.

System (56), the corresponding value function is:

$$J_N = E\left\{\sum_{k=0}^{N} x_k'Qx_k + \sum_{k=d}^{N} u_{k-d}'Ru_{k-d} + x_{N+1}'P_{N+1}x_{N+1}\right\} \quad (57)$$

Q、R、P_{N+1} is half positive definite symmetric matrix with appropriate dimension.

This problem need to look for F_{k-d-1} measurable control u_{k-d}, make J_N minimize.

Based on literature [42], define the following Riccati equation:

$$P_k^1 = A'P_{k+1}^1A + \sigma^2\bar{A}'P_{k+1}^1\bar{A} + A'P_{k+1}^{d+1}A + Q \quad (58)$$

$$P_k^2 = -M_k'\gamma_k^{-1}M_k \quad (59)$$

$$P_k^i = A'P_{k+1}^{i-1}A, i = 3,...,d+1 \quad (60)$$

Terminal condition is satisfied $P_{N+1}^1 = P_{N+1}$, and $P_{N+1}^i = 0, 2 \le i \le d+1$. γ_k, M_k satisfied:

$$\gamma_k = \sum_{j=1}^{d+1} B'P_{k+1}^jB + \sigma^2\bar{B}'P_{k+1}^1\bar{B} + R \quad (61)$$

$$M_k = \sum_{j=1}^{d+1} B'P_{k+1}^jA + \sigma^2\bar{B}'P_{k+1}^1\bar{A} \quad (62)$$

In order to insure the solvability of problem, hypothesis is as follows:

Hypothesis 1 For (58) - (62), assume that the coefficient matrix of the value function (57) meet γ_k reversible, $k = d, d+1, ..., N$

Theorem 4.1 Under the condition of the above hypothesis, the discrete control input with time-delay system (56), there is only the optimal controller makes the corresponding objective function (57) minimize, and the optimal controller is represented as the following form:

$$u_k = -\gamma_{k+d}^{-1}(M_{k+d}\hat{x}_{k+d|k} + r_{k+d}), 0 \le k \le N-d \quad (63)$$

Among them $\hat{x}_{k+d|k}$ is forecast about using the state of the moment k and history input $u_{k-1},...,u_{k-d}$ for the state x_{k+d} of the moment $k+d$, namely:

$$\hat{x}_{k+d|k} = E[x_{k+d}|F_{k-1}] = A^dx_k + \sum_{i=1}^{d} A^{i-1}Bu_{k-i} \quad (64)$$

And r_k of (63) meet the coupling relationship as follows:

$$r_k = \sum_{j=1}^{d+1} B'P_{k+1}^jf + \sigma^2\bar{B}'P_{k+1}^1\bar{f} + B'h_{k+1} \quad (65)$$

$$h_k = -M_k'\gamma_k^{-1}r_k + \sum_{j=1}^{d+1} A'P_{k+1}'f + \sigma^2\bar{A}'P_{k+1}^1\bar{f} + A'h_{k+1} \quad (66)$$

4.4. Terminal Conditions to Meet

$$h_{N+1} = 0, r_N = B'P_{N+1}f + \sigma^2\bar{B}'P_{N+1}\bar{f} \quad (67)$$

Along with the state λ_k and the system state satisfying relationships:

$$\lambda_k = P_{k+1}^1x_{k+1} + \sum_{i=2}^{d+1} P_{k+1}^i\hat{x}_{k+1|k-d+i-1} + h_{k+1} \quad (68)$$

Proof: According to lemma 4.1 established stochastic maximum principle, can get the following relationship

$$\lambda_{k-1} = E[(A+w_k\bar{A})'\lambda_k|F_{k-1}] + Qx_k, k = 0,...,N \quad (69)$$

$$0 = E[(B+w_k\bar{B})'\lambda_k|F_{k-d-1}] + Ru_{k-d}, k = d,...,N \quad (70)$$

And terminal conditions to meet: $\lambda_N = P_{N+1}x_{N+1}$

Using mathematical induction, when $k=N$, by the equilibrium condition (70) can get

$$0=E[(B+w_N\bar{B})'P_{N+1}x_{N+1}|F_{N-d-1}]+Ru_{N-d}$$

$$=E\left\{(B+w_N\bar{B})'P_{N+1}[(A+w_N\bar{A})x_N+(B+w_N\bar{B})u_{N-d}+(f+w_N\bar{f})]|F_{N-d-1}\right\}+Ru_{N-d}$$

$$=(B'P_{N+1}A+\sigma^2\bar{B}'P_{N+1}\bar{A})\hat{x}_{N|N-d}+(R+B'P_{N+1}B+\sigma^2\bar{B}'P_{N+1}\bar{B})u_{N-d}$$

$$+B'P_{N+1}f+\sigma^2\bar{B}'P_{N+1}\bar{f}$$

Notice the type (61), and by the previous assumptions γ_N is reversible, therefore, u_{N-d} can be represented as:

$$u_{N-d}=-\gamma_N^{-1}(M_N\hat{x}_{N|N-d}+r_N) \tag{71}$$

Notice $P_{N+1}^i=0, 2\le i\le d+1$, can get γ_N and M_N by (61) (62) given respectively, $\hat{x}_{N|N-d}$ satisfied (64), and r_N satisfied (67).

By the adjoint equation (69) can be derived:

$$\lambda_{N-1}=E\{(A+w_N\bar{A})'P_{N+1}[A+w_N\bar{A}x_N+(B+w_N\bar{B})u_{N-d}$$

$$+(f+w_N\bar{f})]F_{N-1}\}+Qx_N$$

$$=(A'P_{N+1}A+\sigma^2\bar{A}'P_{N+1}\bar{A}+Q)x_N+(A'P_{N+1}B+$$

$$\sigma^2\bar{A}'P_{N+1}\bar{B})u_{N-d}+(A'P_{N+1}f+\sigma^2\bar{A}'P_{N+1}\bar{f})$$

$$=(A'P_{N+1}A+\sigma^2\bar{A}'P_{N+1}\bar{A}+Q)x_N-(A'P_{N+1}B+$$

$$\sigma^2\bar{A}'P_{N+1}\bar{B})\gamma_N^{-1}(M_N\hat{x}_{N|N-d}+r_N)+(A'P_{N+1}f+\sigma^2\bar{A}'P_{N+1}\bar{f})$$

$$=P_N^1x_N+\sum_{i=2}^{d+1}P_N^i\hat{x}_{N|N-d+i-2}+h_N$$

P_N^1, P_N^1 Satisfied (58) - (60) respectively, h_N by (66) given.

To take advantage of induction, take any $0\le l\le N-d$, for giving $k\ge l+1$, assume the following set up:

(1) λ_{k-1} meet the relationship (68), P_k^1, P_k^i $(2\le i\le d+1)$ satisfied (58)-(60), h_k satisfied (66);

(2) The optimal controller u_{k-d} by (63) given, r_k satisfied (65).

To proof under the above assumptions $k=l$ are also established.

By the above assumption λ_l and the system state to satisfy the following relations:

$$\lambda_l=P_{l+1}^1x_l+\sum_{i=2}^{d+1}P_{l+1}^i\hat{x}_{l+1|l-d+i-2}+h_{l+1}$$

$$=P_{l+1}^1(A+w_l\bar{A})x_l+P_{l+1}^1(B+w_l\bar{B})u_{l-d}+P_{l+1}^1(f+w_l\bar{f})$$

$$+\sum_{i=2}^{d+1}P_{l+1}^i(A\hat{x}_{l|l-d+i-1}+Bu_{l-d}+f_l)+h_{l+1}$$

$$= P_{l+1}^{1}(A + w_l\bar{A})x_l \sum_{i=2}^{d+1} P_{l+1}^{i}A\hat{x}_{l|l-d+i-1} + [P_{l+1}^{1}(B + w_l\bar{B}) + \sum_{i=2}^{d+1} P_{l+1}^{i}B]u_{l-d}$$

$$+P_{l+1}^{1}(f + w_l\bar{f}) + \sum_{i=2}^{d+1} P_{l+1}^{i}f + h_{l+1} \tag{72}$$

Put (72) into (70),

$$0 = E\{(B + w_l\bar{B})'P_{l+1}^{1}(A + w_l\bar{A})x_l + \sum_{i=2}^{d+1}(B + w_l\bar{B})'P_{l+1}^{i}A\hat{x}_{l|l-d+i-1}$$

$$+[(B + w_l\bar{B})'P_{l+1}^{1}(B + w_l\bar{B}) + \sum_{i=2}^{d+1}(B + w_l\bar{B})'P_{l+1}^{i}B]u_{l-d}$$

$$+(B + w_l\bar{B})'[P_{l+1}^{1}(f + w_l\bar{f}) + \sum_{i=2}^{d+1}P_{l+1}^{i}f + h_{l+1}]|F_{l-d-1}\} + Ru_{l-d}$$

$$= E\left\{(B'P_{l+1}^{1}A + \sigma^2\bar{B}'P_{l+1}^{1}\bar{A})x_l + \sum_{i=2}^{d+1}B'P_{l+1}^{i}A\hat{x}_{l|l-d+i-1}|F_{l-d-1}\right\}$$

$$+(B'P_{l+1}^{1}B + \sigma^2\bar{B}'P_{l+1}^{1}\bar{B} + \sum_{i=2}^{d+1}B'P_{l+1}^{i}B)u_{l-d} + Ru_{l-d}$$

$$+B'P_{l+1}^{1}f + \sigma^2\bar{B}'P_{l+1}^{1}\bar{f} + \sum_{i=2}^{d+1}B'P_{l+1}^{i}f + B'h_{l+1} \tag{73}$$

By the previous assumption γ_l is reversible, therefore the (60) (61) shows that u_{l-d} can be represented as:

$$u_{l-d} = -\gamma_l^{-1}(M_l\hat{x}_{l|l-d} + r_l) \tag{74}$$

Namely the optimal controller (63) was established for K.
Then prove that λ_{l-1} satisfy (68), according (69) (72) and (73), we can get:

$$\lambda_{l-1} = E\{[(A + w_l\bar{A})'P_{l+1}^{1}(A + w_l\bar{A}) + (A + w_l\bar{A})'P_{l+1}^{d+1}A]x_l$$

$$-[(A + w_l\bar{A})'P_{l+1}^{1}(B + w_l\bar{B}) + \sum_{i=2}^{d+1}(A + w_l\bar{A})'P_{l+1}^{i}B]\gamma_l^{-1}(M_l\hat{x}_{l|l-d} + r_l)$$

$$+\sum_{i=2}^{d+1}(A + w_l\bar{A})'P_{l+1}^{i}A\hat{x}_{l|l-d+i-1}|F_{l-1}\} + Qx_l + A'P_{l+1}^{1}f$$

$$+\sigma^2\bar{A}'P_{l+1}^{1}\bar{f} + \sum_{i=2}^{d+1}A'P_{l+1}^{i}f + A'h_{l+1}$$

$$= (Q + (A)'P_{l+1}^{1}A + \bar{A}'P_{l+1}^{1}\bar{A} + A'P_{l+1}^{d+1}A)x_l + \sum_{i=3}^{d+1}(A)'P_{l+1}^{i-1}A\hat{x}_{l|l-d+i-2}$$

$$-[(A + w_l\bar{A})'P_{l+1}^{1}(B + w_l\bar{B}) + \sum_{i=2}^{d+1}(A + w_l\bar{A})'P_{l+1}^{i}B]\gamma_l^{-1}(M_l\hat{x}_{l|l-d} + r_l)$$

$$+A'P^1_{l+1}f+\sigma^2\overline{A}'P^1_{l+1}\overline{f}+\sum_{i=2}^{d+1}A'P^i_{l+1}f+A'h_{l+1} \tag{75}$$

In addition, because there is the following relationship:

$$E[(A+w_l\overline{A})'P^{d+1}_{l+1}A]=A'P^{d+1}_{l+1}A=\left(A'\right)^2P^d_{l+2}=\ldots=\left(A'\right)^dP^2_{l+d}A^d$$

$$M'_l=A'P_{l+1}B+\sigma^2\overline{A}'P^1_{l+1}\overline{B}+\sum_{j=2}^{d+1}A'P^j_{l+1}B$$

Therefore, (74) can be represented as:

$$\lambda_{l-1}=P^1_lx_l+\sum_{i=2}^{d+1}P^i_l\hat{x}_{l|l-d+i-2}+h_l \tag{76}$$

So h_l satisfied:

$$h_l=-M'_l\gamma_l^{-1}r_l+\sigma^2\overline{A}'P^1_{k+1}g_l+\sum_{j=1}^{d+1}A'P^1_{l+1}f_l+A'h_{l+1}$$

So it has proved that the conclusion of theorem to $k=l$ also set up. According to the mathematical induction, q.e.d.

4.5. Solution of the Model (39)

Now back to the time-delay discrete form about solving auxiliary problem (39) of the mean - variance model, contrasting the model with time-delays discrete LQ control problem, Find the model is a special case of the time- delay general discrete LQ problems.

$$A_k=1+r^0_k,\overline{A}=0,B_k=R^i_k=r^i_k-r^0_k$$

$$\overline{B}=D_k=\sum_{j=1}^{m}\sum_{i=1}^{m}\sigma_{ij},f=\gamma r^0_k,\overline{f}=0.$$

Likewise, applying the maximum principle can get the following conclusion:

$$\lambda_N=\mu x_{N+1}$$

$$\lambda_{k-1}=E[A'_k\lambda_k|F_{k-1}],k=0,\ldots,N \tag{77}$$

$$0=E\left\{(B_k+w_kD_k)'\lambda_k|F_{k-d-1}\right\},k=d,\ldots,N \tag{78}$$

Applying in front of the conclusion, can get the optimal portfolio proportion directly:

$$u_k=-\gamma^{-1}_{k+d}(M_{k+d}\hat{y}_{k+d|k}+r_{k+d}) \tag{79}$$

Among them

$$\hat{y}_{k+d|k}=E[y_{k+d}|F_{k-1}]=A^dy_k+\sum_{i=1}^{d}A^{i-1}Bu_{k-i}$$

$$y_k=x_k-\gamma,r_k=B'P_{k+1}f_k+B'h_{k+1}$$

$$\lambda_{k-1}=P'_ky_k+\sum_{i=2}^{d+1}P^i_k\hat{y}_{k|k-d+i-2}+h_k$$

$$h_k=M'_k\gamma^{-1}_kr_k+A'P_{k+1}f_k,h_{N+1}=0$$

4.6. The Example Analysis

Assume that an investor has initial wealth of one million yuan, investing in a risk-free securities and a risk stock respectively; annual return of risk-free securities is $r^0_k=6\%$, annual return of risk stock is $r^1_k=16\%$, The standard deviation for $\sigma=20\%$, Time delay $d=2$, and $u_{-1}=1,u_{-2}=2$, Investors do four investment strategy adjustment, i.e. $N=4,\mu=2$, then $P_{N+1}=1,1+r^0_k=1.06$, In addition, the white noise as Gaussian white noise $w\sim N(0,1)$. Use theorem 4.1 to calculate directly:

$P^1_4=1.1236$	$P^1_3=1.2625$	$P^1_2=1.1348$
$P^2_4=-0.2247$	$P^2_3=-0.1685$	$P^2_2=-0.1351$
$P^3_4=0$	$P^3_3=-0.2525$	$P^3_2=-0.1893$
$\gamma_2=0.0589$	$\gamma_3=0.0539$	$\gamma_4=0.0500$
$M_2=0.0892$	$M_3=0.0953$	$M_4=0.1060$

Due to $\gamma_i\succ0(i=2,3,4)$, therefore, by theorem 4.1, exist the optimal solution, and (70) can get the optimal control for:

$$u_0 = -1.5144\hat{y}_{2|0} - \frac{r2}{0.0589}$$

$$u_1 = -1.7681\hat{y}_{3|1} - \frac{r3}{0.0539}$$

$$u_2 = -2.1200\hat{y}_{4|2} - \frac{r4}{0.0500}$$

5. Conclusion

China's capital market for many of China's listed companies play a role of the financing to raise asset allocation, and also make the public investment fans involved in the economic development, share the achievements of economic development. Especially with the popularity of Internet +, more and more online financial products such as the balance of the treasure, earnings treasure in endlessly, make more and more investors change their idea of financial management, investment diversification. Due to capital market increasingly complex, the simple qualitative analysis can't meet the needs of investors and quantitative investment analysis has become the inevitable choice of investors, and the method is based on mathematics, statistics and information technology tools, through the establishment of the corresponding portfolio optimization model, make a comprehensive analysis of the market macro data, enterprise financial position and trading, seeking reliable stability probability distribution, according to the change of data in a timely manner to adjust investment strategy. It turns out, Quantitative investment analysis to intensification of operation and management, are often good at capturing and grasp the fleeting investment opportunities in the market, achieve better long-term stable returns. With the rapid development of economy in our country, the capital market gradually perfect, the financial product is increasing, especially some risks such as balance of treasure of wealth management products, make the broad masses of investors will not put money only one basket, will be carried out in a certain portfolio investment, in order to obtain greater benefits. The mean - variance model is one of the basic calculation model of portfolio strategy, previous researchers did not consider delay problems, on the basis of existing research, this paper established the delay discrete mean - variance model, stochastic optimal control with the aid of LQ problem, to get the analytical solution of the optimal portfolio, further, also can consider to delay the continuous mean - variance portfolio model. Although the significance of this research is more important, the result is satisfactory, but its research conclusion and the actual still has a gap, there are still many aspects need further research, such as:

1) On the basis of solving the optimal strategy, to further study the effective frontier of portfolio, analysis the relationship between the expected return and risk; 2) Considering it contains transaction fees and consumption in the model; 3) considering investing object contains options, etc.

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