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I-Statistically Pre-cauchy Triple Sequences of Fuzzy Real Numbers

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Abstract: In this article, using Orlicz function, the concept of I-statistically pre-Cauchy sequence of fuzzy real numbers having multiplicity greater than two is introduced. A necessary and sufficient condition for a bounded triple sequence of fuzzy real numbers to be I-statistically Cauchy is established. It is also shown that an I-statistically convergent triple sequence of fuzzy numbers is I-statistically pre-Cauchy.

Keywords: Ideal, Filter, Triple Sequence of Fuzzy Numbers, Statistical Convergence, Ideal Convergence, I-Statistically Convergence, I-Statistically Pre-cauchy, Orlicz Function

1. Introduction

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. Fuzzy set theory has become an area of active area of research in science and engineering for the last 46 years. The concept of fuzzy set is first introduced by Zadeh [37] in 1965. Later on, the sequence of fuzzy numbers is discussed by several mathematicians such as Matloka [18], Nanda [20], Savas [28] and many others.

The notion of statistical convergence was first introduced by Fast [11]. After then it was studied by many researchers like Šalát [25], Fridy [12], Connor [2], Maddox [17], Kwon [16], Savas [27]. Different classes of statistically convergent sequences were introduced and investigated by Tripathy and Sen [35], Tripathy and Sarma [34] etc. Móricz [19] extended statistical convergence from single to multiple real sequences. Nuray and Savas [22] first defined the concepts of statistical convergence and statistically Cauchy for sequences of fuzzy numbers. The notion of statistically pre-Cauchy for real sequences was introduced by Connor, Fridy and Kline [3]. More works on statistically pre-Cauchy sequences are found in Khan and Lohani [14], Dutta [8], Dutta and Tripathy [7], Das *et al*. [5] etc.

Agnew [1] studied the summability theory of multiple sequences and obtained certain theorems which have already been proved for double sequences by the author himself. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner *et al*. [23], Sahiner and Tripathy [24]. Recently Savas and Esi [30] have introduced statistical convergence of triple sequences on probabilistic normed space. Later on, Esi [9] have introduced statistical convergence of triple sequences in topological groups. Some more works on triple sequences are found on Kumar *et al.* [15], Esi [10], Dutta *et al.* [6], Tripathy and Dutta [32], Tripathy and Goswami [33], Nath and Roy [21] etc.

The idea of statistical convergence is extended to *I*-convergence in case of real's by using the notion of ideals of *N.* Kostyrko, Šalát and Wilczyński [13] introduced the concept of ideal convergence for single sequences in 2000-2001. Later on it was further developed by Šalát *et al*. [26], Das *et al.* [4], Sen and Roy [31] and many others. Savas and Das [29] used ideals to introduce the concept of I-statistical convergence for real's which have extended the notion of statistical convergence. The notion of I-statistically pre-Cauchy sequences are also introduced by them. In the present article, we have extended these results to introduce the concept of I-statistically pre-Cauchy sequences for the triple sequence of fuzzy real numbers.

A fuzzy real number on R is a mapping $X : R \to L(=[0,1])$ associating each real number $t \in R$ with its grade of membership X (t). Every real number r can be expressed as a fuzzy real number $\bar{r}$ as follows:

$$\bar{r}\ (t) = \begin{cases} 1 & if \quad t = r \\ 0 & otherwise \end{cases}$$

The α-level set of a fuzzy real number X, $0 < \alpha \le 1$, denoted by $[X]^{\alpha}$ is defined as

$$[X]^{\alpha} = \{t \in R : X(t) \ge \alpha\}.$$

A fuzzy real number X is called convex if $X(t) \ge X(s) \wedge X(r) =$ min $(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called normal. A fuzzy real number X is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}[0, a+\varepsilon))$, for all $a \in L$ is open in the usual topology of R. The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(L)$. The additive identity and multiplicative identity in $R(L)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Let D be the set of all closed bounded intervals $X = \left[X^L, X^R\right]$ on the real line R. Then

$X \le Y$ if and only if $X^L \le Y^L$ and $X^R \le Y^R$. Also let $d(X,Y)$ = max ($|X^L$-$X^R|$, $|Y^L$-$Y^R|$).

Then (D, d) is a complete metric space.

Let $\bar{d} : R(L) \times R(L) \to R$ be defined by $\bar{d}(X,Y) = \sup\limits_{0 \le \alpha \le 1} d([X]^{\alpha}, [Y]^{\alpha})$, for $X, Y \in R(L)$.

Then $\bar{d}$ defines a metric on $R(L)$.

2. Preliminaries and Background

In this section, some notations and basic definitions which will be used in this article are recalled.

A triple sequence can be defined as a function $x : N \times N \times N \to R(C)$, where N, R and C denote the sets of natural numbers, real numbers and complex numbers respectively.

The notion of statistical convergence for triplesequences depends on the density of the subsets of $N \times N \times N$. A subset E of $N \times N \times N$ is said to have density or asymptotic density $\delta_3(E)$, if the limit given by $\delta_3(E) = \lim\limits_{p,q,r \to \infty} \sum\limits_{n=1}^{p} \sum\limits_{k=1}^{q} \sum\limits_{l=1}^{r} \chi_E(n,k,l)$ exists, where χ_E is the characteristic function of E.

Obviously $\delta_3(E^c) = \rho(N \times N \times N - E) = 1 - \delta_3(E)$. The notion of Ideal convergence depends on the structure of the ideal I of the subset of the set of natural numbers N.

Let X be a non empty set. A non-void class $I \subseteq 2^X$ (power set of X) is said to be an ideal if I is additive and hereditary, *i. e.* if I satisfies the following conditions:

(i) $A, B \in I \Rightarrow A \cup B \in I$ and

(ii) $A \in I$ *and* $B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a filter on X if

(*i*) $\emptyset \notin F$ (*ii*) $A, B \in F \Rightarrow A \cap B \in F$ and (*iii*) $A \in F$ and $A \subseteq B \Rightarrow B \in F$.

For any ideal I, there is a filter F (I) given by $F(I) = \{ K \subseteq N : N \setminus K \in I \}$.

An ideal $I \subseteq 2^X$ is said to be non-trivial if $I \neq \emptyset$ and X $\notin I$. The details about the ideals of $2^{N \times N}$ are introduced and investigated by Tripathy and Tripathy [36]. Throughout, the ideals of $2^{N \times N \times N}$ will be denoted by I_3. .

Example 2. 1. Let $I_3(\rho) \subset 2^{N \times N \times N}$ i. e. the class of all subsets of $N \times N \times N$ of zero natural density.

Then $I_3(\rho)$ is an ideal of $2^{N \times N \times N}$.

The scope for the studies on sequence spaces was extended by using the concept of Orlicz function. The study of Orlicz sequence spaces was initiated with certain specific purpose in Banach space theory.

An Orlicz function M is a function $M : [0,\infty) \to [0,\infty)$ such that it is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [14] used the idea of Orlicz function to construct the sequence space,

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which becomes a Banach space, with the norm $\|x\| = \inf\left\{ \rho > 0 : \sum\limits_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$.

The space ℓ_M is closely related to the space ℓ^p, which is an Orlicz sequence space with $M(x) = x^p$, for $1 \le p < \infty$. If M is an Orlicz function, then $M(0) = 0$, and $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$. An Orlicz function may be bounded or unbounded. For example,

$M(x) = x^p$, $(0 < p \le 1)$ is unbounded and $M(x) = \dfrac{x}{x+1}$ is bounded.

Studies on some important classes of Orlicz sequences from different point of view are found in Parashar and Choudhary [21], Tripathy and Sarma [33], Sen and Roy [27] etc.

Throughout the article $(w^F)_3$ and $(\ell^F)_3$ denote the spaces of all and bounded triple sequences of fuzzy numbers respectively.

A fuzzy real-valued triple sequence $X = \left\langle X_{ijk} \right\rangle$ is a triple infinite array of fuzzy real numbers X_{ijk} for all $i, j, k \in N$ and is denoted by $\langle X_{ijk} \rangle$ where $X_{ijk} \in R(L)$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be statistically convergent to the fuzzy real number L, if for all $\varepsilon > 0$,

$$\lim_{m,n,l\to\infty} \frac{1}{mnl} \left| \{(i,j,k) : \bar{d}(X_{ijk}, L) \ge \varepsilon, i \le m; j \le n; k \le l\} \right| = 0,$$

where $|.|$ indicate the cardinality of the enclosed set and we write $stat_3 - \lim X_{ijk} = L$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be I-convergent to the fuzzy real number L, if for each $\varepsilon > 0$, the set $\left\{(i,j,k) \in N \times N \times N : \bar{d}(X_{ijk}, L) \ge \varepsilon \right\} \in I_3$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be bounded if $\sup_{i,j,k} \bar{d}(X_{ijk}, \bar{0}) < \infty$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be I-statistically convergent to the fuzzy real number L, if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(m,n,l) \in N \times N \times N : \frac{1}{mnl} \left| \{ i \le m; j \le n; k \le l : \bar{d}(X_{ijk}, L) \ge \varepsilon \} \right| \ge \delta \right\} \in I_3.$$

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be I-statistically pre-Cauchy if, for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(m,n,l) \in N \times N \times N : \frac{1}{m^2 n^2 l^2} \left| \{(i,j,k) : \bar{d}(X_{ijk}, X_{pqr}) \ge \varepsilon, i, p \le m; j, q \le n; k, r \le l\} \right| \ge \delta \right\} \in I_3.$$

3. Main Results

Using Orlicz function, a necessary and sufficient condition for a bounded triple sequence of fuzzy real numbers to be *I*-statistically pre-Cauchy is derived in the following theorem.

Theorem 3. 1. Let M be a bounded Orlicz function. Then a triple sequence of fuzzy numbers $X = \langle X_{ijk} \rangle$ is *I*-statistically pre-Cauchy if and only if

$$I - \lim_{m,n,l\to\infty} \frac{1}{m^2 n^2 l^2} \sum_{i,p\le m} \sum_{j,q\le n} \sum_{k,r\le l} M\left(\frac{\bar{d}(X_{ijk}, X_{pqr})}{\zeta} \right) = 0, \text{ for some } \zeta > 0.$$

Proof. Let

$$I - \lim_{m,n,l\to\infty} \frac{1}{m^2 n^2 l^2} \sum_{i,p\le m} \sum_{j,q\le n} \sum_{k,r\le l} M\left(\frac{\bar{d}(X_{ijk}, X_{pqr})}{\zeta} \right) = 0, \text{ for some } \zeta > 0. \tag{1}$$

For each $\varepsilon > 0, \zeta > 0$ and $m, n, l \in N$,

$$\frac{1}{m^2 n^2 l^2} \sum_{i,p\le m} \sum_{j,q\le n} \sum_{k,r\le l} M\left(\frac{\bar{d}(X_{ijk}, X_{pqr})}{\zeta} \right)$$

$$= \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p\le m \\ \bar{d}(X_{nkl}, X_{pqr}) < \varepsilon}} \sum_{j,q\le n} \sum_{k,r\le l} M\left(\frac{\bar{d}(X_{ijk}, X_{pqr})}{\zeta} \right) + \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p\le m \\ \bar{d}(X_{nkl}, X_{pqr}) \ge \varepsilon}} \sum_{j,q\le n} \sum_{k,r\le l} M\left(\frac{\bar{d}(X_{ijk}, X_{pqr})}{\zeta} \right)$$

$$\ge \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p\le m \\ \bar{d}(X_{nkl}, X_{pqr}) \ge \varepsilon}} \sum_{j,q\le n} \sum_{k,r\le l} M\left(\frac{\bar{d}(X_{ijk}, X_{pqr})}{\zeta} \right) \ge M(\varepsilon)\left(\frac{1}{m^2 n^2 l^2} \left| \{(i,j,k) : \bar{d}(X_{ijk}, X_{pqr}) \ge \varepsilon, i, p \le m; j, n \le n; k, r \le l\} \right| \right).$$

Therefore for any $\delta > 0$ and using (1)

$$\left\{(m,n,l) \in N \times N \times N : \frac{1}{m^2 n^2 l^2} \left| \{(i,j,k) : \bar{d}(X_{ijk}, X_{pqr}) \ge \varepsilon, i, p \le m; j, n \le n; k, r \le l\} \right| \ge \delta \right\}$$

$$\subset \left\{ (m,n,l) \in N \times N \times N : \frac{1}{m^2 n^2 l^2} \sum_{i,p \le m} \sum_{j,q \le n} \sum_{k,r \le l} \left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr}) \right] \ge \delta\varepsilon \right\} \in I_3.$$

Hence X is *I*-statistically pre-Cauchy.

Conversely let $X = \langle X_{ijk} \rangle$ be *I*-statistically pre-Cauchy and M be a bounded Orlicz function.

For a given $\varepsilon > 0$, choosing $\delta > 0$ such that $M(\delta) < \frac{\varepsilon}{2}$ and since M is bounded, so there exists a positive integer K such that $M(x) < \frac{K}{2}$, for all $x \ge 0$.

Then for any $\varepsilon > 0$, and for each $m, n, l \in N$,

$$\frac{1}{m^2 n^2 l^2} \sum_{i,p \le m} \sum_{j,q \le n} \sum_{k,r \le l} M \left(\frac{\bar{d}\ (X_{ijk}\ ,\ X_{pqr})}{\zeta} \right)$$

$$= \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p \le m \\ \bar{d}(X_{nkl}, X_{pqr}) < \varepsilon}} \sum_{j,q \le n} \sum_{k,r \le l} M \left(\frac{\bar{d}\ (X_{ijk}\ ,\ X_{pqr})}{\zeta} \right) + \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p \le m \\ \bar{d}(X_{nkl}, X_{pqr}) \ge \varepsilon}} \sum_{j,q \le n} \sum_{k,r \le l} M \left(\frac{\bar{d}\ (X_{ijk}\ ,\ X_{pqr})}{\zeta} \right)$$

$$\le M(\delta) + \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p \le m \\ \bar{d}(X_{nkl}, X_{pqr}) \ge \varepsilon}} \sum_{j,q \le n} \sum_{k,r \le l} M \left(\frac{\bar{d}\ (X_{ijk}\ ,\ X_{pqr})}{\zeta} \right)$$

$$\le \frac{\varepsilon}{2} + \frac{K}{2} \left(\frac{1}{m^2 n^2 l^2} \left| \{(i,j,k) : \bar{d}(X_{ijk}, X_{pqr}) \ge \varepsilon, i, p \le m; j, n \le n; k, r \le l\} \right| \right). \quad (2)$$

Since X is *I*-statistically pre-Cauchy, the R. H. S. in (2) less than ε for all $m, n, l \in N$. Hence, for some $\zeta > 0$,

$$I - \lim_{m,n,l \to \infty} \frac{1}{m^2 n^2 l^2} \sum_{i,p \le m} \sum_{j,q \le n} \sum_{k,r \le l} M \left(\frac{\bar{d}\ (X_{ijk}\ ,\ X_{pqr})}{\zeta} \right) = 0,$$

The following theorem gives a necessary and sufficient condition for a bounded triple sequence of fuzzy real numbers to be *I*-statistically pre-Cauchy.

Theorem 3. 2. Let $X = \langle X_{ijk} \rangle$ be a bounded triple sequence of fuzzy numbers. Then $X = \langle X_{ijk} \rangle$ is *I*-statistically pre-Cauchy if and only if $I - \lim \frac{1}{m^2 n^2 l^2} \sum_{i,p \le m} \sum_{j,q \le n} \sum_{k,r \le l} \left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr}) \right] = 0.$

Proof. Let

$$I - \lim \frac{1}{m^2 n^2 l^2} \sum_{i,p \le m} \sum_{j,q \le n} \sum_{k,r \le l} \left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr}) \right] = 0. \quad (3)$$

For each $\varepsilon > 0$, and $m, n, l \in N$,

$$\frac{1}{m^2 n^2 l^2} \sum_{i,p \le m} \sum_{j,q \le n} \sum_{k,r \le l} \left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr}) \right] = \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p \le m \\ \bar{d}(X_{nkl}, X_{pqr}) < \varepsilon}} \sum_{j,q \le n} \sum_{k,r \le l} \left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr}) \right] + \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p \le m \\ \bar{d}(X_{nkl}, X_{pqr}) \ge \varepsilon}} \sum_{j,q \le n} \sum_{k,r \le l} \left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr}) \right]$$

$$\ge \frac{1}{m^2 n^2 l^2} \sum_{\substack{i,p \le m \\ \bar{d}(X_{nkl}, X_{pqr}) \ge \varepsilon}} \sum_{j,q \le n} \sum_{k,r \le l} \left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr}) \right] \ge \varepsilon \left(\frac{1}{m^2 n^2 l^2} \left| \{(i,j,k) : \bar{d}(X_{ijk}, X_{pqr}) \ge \varepsilon, i, p \le m; j, q \le n; k, r \le l\} \right| \right).$$

Therefore for any $\delta > 0$, and using (3)

$$\left\{(m,n,l)\in N\times N\times N:\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\geq\varepsilon, i,p\leq m; j,q\leq n; k,r\leq l\}\right|\geq\delta\right\}$$

$$\subset\left\{(m,n,l)\in N\times N\times N:\frac{1}{m^2n^2l^2}\sum_{i,p\leq m}\sum_{j,q\leq n}\sum_{k,r\leq l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]\geq\delta\varepsilon\right\}\in I_3.$$

Hence X is I-statistically pre-Cauchy.

Conversely let X be I-statistically pre-Cauchy. Since $X=\langle X_{ijk}\rangle$ is bounded, so there exists $M>0$ such that $\bar{d}(X_{ijk},\bar{0})<M$, for all $i,j,k\in N$.

The for each $\varepsilon>0$, and $m,n,l\in N$,

$$\frac{1}{m^2n^2l^2}\sum_{i,p\leq m}\sum_{j,q\leq n}\sum_{k,r\leq l}\sum_{k,r\leq l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]$$

$$=\frac{1}{m^2n^2l^2}\sum_{\substack{i,p\leq m\\ \bar{d}(X_{nkl},L)<\frac{\varepsilon}{2}}}\sum_{j,q\leq n}\sum_{k,r\leq l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]+\frac{1}{m^2n^2l^2}\sum_{\substack{i,p\leq m\\ \bar{d}(X_{nkl},L)\geq\frac{\varepsilon}{2}}}\sum_{j,q\leq n}\sum_{k,r\leq l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]$$

$$\leq\frac{1}{m^2n^2l^2}m^2n^2l^2\frac{\varepsilon}{2}+2M\left(\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\geq\frac{\varepsilon}{2}, i\leq m; j\leq n; k\leq l\}\right|\right).$$

Since X is I-statistically pre-Cauchy, for any $\delta>0$,

$$A=\left\{(m,n,l)\in N\times N\times N:\ \frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\geq\frac{\varepsilon}{2}, i\leq m; j\leq n; k\leq l\}\right|\geq\delta\right\}\in I_3.$$

Then for all $(m,n,l)\in A^c$,

$$\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\geq\frac{\varepsilon}{2}, i\leq m; j\leq n; k\leq l\}\right|<\delta.$$

$$\therefore\frac{1}{m^2n^2l^2}\sum_{i,p\leq m}\sum_{j,q\leq n}\sum_{k,r\leq l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]\leq\frac{\varepsilon}{2}+2M\delta.$$

Let $\delta_1>0,$ be chosen such that $\frac{\varepsilon}{2}+2M\delta<\delta_1.$

Then for all $(m,n,l)\in A^c$,

$$\frac{1}{m^2n^2l^2}\sum_{i,p\leq m}\sum_{j,q\leq n}\sum_{k,r\leq l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]\leq\delta_1.$$

$$\therefore\left\{(m,n,l)\in N\times N\times N:\frac{1}{m^2n^2l^2}\sum_{i,p\leq m}\sum_{j,q\leq n}\sum_{k,r\leq l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]\geq\delta_1\right\}\subset A\in I_3.$$

Hence $I-\lim\frac{1}{m^2n^2l^2}\sum_{i,p\leq m}\sum_{j,q\leq n}\sum_{k,r\leq l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]=0.$

Theorem 3. 3. Let $X=\langle X_{ijk}\rangle$ be abounded triple sequence of fuzzy number and M be a bounded Orlicz function. Then $I-\lim_{m,n,l\to\infty}\frac{1}{m^2n^2l^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right]=0$ if and only if $I-\lim_{m,n,l\to\infty}\frac{1}{m^2n^2l^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[M\left(\frac{\bar{d}\,(X_{ijk}\ ,\ X_{pqr})}{\zeta}\right)\right]=0,$ for some $\zeta>0.$

Proof. Let $A = \sup_{i,j,k} \bar{d}(X_{ijk}, \bar{0})$ and define $M(x) = (1+3A)\dfrac{x}{x+1}$

Then for some $\zeta > 0$, $M\left(\dfrac{\bar{d}\,(X_{ijk}\ ,\ X_{pqr})}{\zeta}\right) \le (1+3A)\bar{d}\,(X_{ijk}\ ,\ X_{pqr})$ and

$$M\left(\frac{\bar{d}\,(X_{ijk}\ ,\ X_{pqr})}{\zeta}\right) = (1+3A)\frac{M(X_{ijk}\ ,\ X_{pqr})}{1+\bar{d}\,(X_{ijk}\ ,\ X_{pqr})} \ge \frac{(1+3A)\bar{d}(X_{ijk}\ ,\ X_{pqr})}{1+\bar{d}\,(X_{ijk}\ ,\ \bar{0})+\bar{d}(X_{pqr},\bar{0})} \ge \frac{(1+3A)\bar{d}(X_{ijk}\ ,\ X_{pqr})}{(1+3A))} = \bar{d}(X_{ijk}\ ,\ X_{pqr})$$

Therefore $I-\lim_{m,n,l\to\infty} \dfrac{1}{m^2n^2l^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right] = 0,$ if and only if

$I-\lim_{m,n,l\to\infty} \dfrac{1}{m^2n^2l^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} M\left(\dfrac{\bar{d}\,(X_{ijk}\ ,\ X_{pqr})}{\zeta}\right) = 0,$ for some $\zeta > 0$.

Theorem 3. 4. Let $X = \langle X_{ijk}\rangle$ be abounded triple sequence of fuzzy numbers. Then $X = \langle X_{ijk}\rangle$ is *I*-statistically pre-Cauchy if and only if $I-\lim \dfrac{1}{m^2n^2l^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\,(X_{ijk}\ ,\ X_{pqr})\right] = 0.$

Proof. Using Theorem 3. 2 and Theorem 3. 3, the desired result can be obtained.

Theorem 3. 5. Let $X = \langle X_{ijk}\rangle$ be a bounded triple sequence of fuzzy numbers. Then $X = \langle X_{ijk}\rangle$ is *I*-statistically convergent to *L*, if and only if $I-\lim \dfrac{1}{mnl}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\,(X_{ijk}\ ,\ L)\right] = 0.$

Proof. Considering $A = \sup_{i,j,k} \bar{d}(X_{ijk}, \bar{0})$ and $M(x) = (1+A+L)\dfrac{x}{x+1}$, then the proof is similar to that of Theorem 3. 4.

Theorem 3. 5. A *I*-statistically convergent fuzzy real-valued triple sequence $\langle X_{ijk}\rangle$ is *I*-statistically pre-Cauchy, but an *I*-statistically pre-Cauchy triple sequence of fuzzy numbers need not be *I*-statistically convergent.

Proof. Let the fuzzy real-valued triple sequence $X = \langle X_{ijk}\rangle$ be I-statistically convergent to L. Then for each $\varepsilon > 0$, and $m,n,l \in N$

$$I-\lim_{m,n,l\to\infty} \frac{1}{mnl}\left|\{(i,j,k) : \bar{d}(X_{ijk}, L) \ge \varepsilon, i \le m; j \le n; k \le l\}\right| = 0.$$

So for any $\varepsilon, \rho > 0$,

let $A = \left\{(m,n,l) \in N\times N\times N : \dfrac{1}{mnl}\left|\{(i,j,k) : \bar{d}(X_{ijk}, L) \ge \varepsilon, i \le m; j \le n; k \le l\}\right| \ge \rho\right\}.$

Then for $(m,n,l) \in A^c$ (complement of *A*),

$$\frac{1}{mnl}\left|\{(i,j,k) : \bar{d}(X_{ijk}, L) \ge \frac{\varepsilon}{2}, i \le m; j \le n; k \le l\}\right| < \rho.$$

$$\Rightarrow \frac{1}{mnl}\left|\{(i,j,k) : \bar{d}(X_{ijk}, L) < \frac{\varepsilon}{2}, i \le m; j \le n; k \le l\}\right| > 1-\rho.$$

Let $A_{mnl} = \left\{(i,j,k) : \bar{d}(X_{ijk}, L) < \varepsilon, i \le m; j \le n; k \le l\right\}.$

Then

$$\frac{1}{mnl}\left|A_{mnl}\right| > 1-\rho \tag{4}$$

So for all $(i,j,k) \in A_{mnl}$

$$\bar{d}(X_{ijk},X_{pqr})\le \bar{d}(X_{ijk},L)+\bar{d}(X_{pqr},L)<\varepsilon \Rightarrow A_{mnl}\times A_{mnl}\subseteq \{(i,j,k):\bar{d}(X_{ijk},X_{pqr})<\varepsilon, i\le m; j\le n; k\le l\}.$$

$$\Rightarrow \frac{|A_{mnl}\times A_{mnl}|}{m^2n^2l^2}=\frac{\left[|A_{mnl}|\right]^2}{m^2n^2l^2}\le \frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})<\varepsilon, i\le m; j\le n; k\le l\}\right|$$

$$\Rightarrow \frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})<\varepsilon, i\le m; j\le n; k\le l\}\right|\ge \left[\frac{|A_{mnl}|}{mnl}\right]^2>(1-\rho)^2, \text{ using (4).}$$

So $\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\varepsilon, i\le m; j\le n; k\le l\}\right|<1-(1-\rho)^2$.

For any given $\rho_1>0$, choosing δ>0 such that $1-(1-\rho)^2<\rho_1$.

Now for all $(m,n,l)\in A^c$

$$\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\varepsilon, i\le m; j\le n; k\le l\}\right|<\rho_1.$$

$$\therefore \left\{(i,j,k)\in N\times N\times N:\ \frac{1}{m^2n^2l^2}\left|(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\varepsilon, i\le m; j\le n; k\le l\right|\ge\rho_1\right\}\subset A\in I_3.$$

$$\therefore I-\lim_{m,n,l\to\infty}\frac{1}{m^2n^2l^2}\left|(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\varepsilon, i\le m; j\le n; k\le l\right|=0.$$

Hence $\langle X_{ijk}\rangle$ is *I*-statistically pre-Cauchy.

But an *I*-statistically pre-Cauchy triple sequence of fuzzy numbers need not be *I*-statistically convergent which follows from the following example.

Example 3. 1. Consider the fuzzy triple sequence $\langle X_{ijk}\rangle$ defined as follows:

$X_{ijk}=\overline{A_{mnl}}$, $A_{mnl}=\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left(\frac{1}{ijk}\right)$, where $(m-1)!<i<m!, (n-1)!<j<n!, (l-1)!<k<l!$.

Then for each $0<\alpha\le 1$, the α-cut of $\langle X_{ijk}\rangle$ is given by, $\left[X_{ijk}\right]^{\alpha}=A_{mnl}$.

Then it can be proved that $\langle X_{ijk}\rangle$ is *I*-statistically pre-Cauchy, but not *I*-statistically convergent.

4. Conclusion

Using Orlicz function, the notion of the *I*-statistically pre-Cauchy sequence of fuzzy real numbers having multiplicity greater than two is introduced. A necessary and sufficient condition for a bounded triple sequence of fuzzy real numbers to be *I*-statistically Cauchy is derived. It is also shown that an *I*-statistically convergent triple sequence of fuzzy numbers is *I*-statistically pre-Cauchy. The introduced notion can be applied for further investigations from different aspects.

References

[1] R. P. Agnew, On summability of multiple sequences, *American Journal of Mathematics*; 1 (4), 62-68, (1934).

[2] J. S. Connor, The statistical and strong *p*-Cesáro convergence of sequences, *Analysis*; 8, 47-63, (1988).

[3] J. S. Connor, J. Fridy, J. Kline, Statistically pre-Cauchy sequences, *Analysis*; 14, 311-317, (1994).

[4] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, *I* and *I** convergence of double Sequences, *Math. Slovaca*; 58 (5), 605-620, (2008).

[5] P. Das, E. Savas, On *I*-statistically pre-Cauchy Sequences, *Taiwanese Journal of Mathematics*; 18 (1), 115-126, (2014).

[6] A. J. Dutta, A. Esi, B. C. Tripathy, Statistically convergence triple sequence spaces defined by Orlicz function, *Journal of Mathematical Analysis*; 4 (2), 16-22, (2013).

[7] A. J. Dutta, B. C. Tripathy, Statistically pre-Cauchy Fuzzy real-valued sequences defined by Orlicz function, *Proyecciones Journal of Mathematics*; 33 (3), 235-243, (2014).

[8] H. Dutta, A Characterization of the Class of statistically pre-Cauchy Double Sequences of Fuzzy Numbers, *Appl. Math. Inf. Sci*.; 7 (4), 1437-1440, (2013).

[9] A. Esi, Statistical convergence of triple sequences in topological groups, Annals of the University of Craiova, *Mathematics and Computer Science Series*; 40 (1), 29-33, (2013).

[10] A. Esi, λ_3-Statistical convergence of triple sequences on probabilistic normed space, *Global Journal of Mathematical Analysis*; 1 (2), 29-36, (2013).

[11] H. Fast, Surla convergence statistique, *Colloq. Math*.; 2, 241-244, (1951).

[12] J. A. Fridy, On statistical convergence, Analysis, 5, 301-313, (1985).

[13] P. Kostyrko, T. Šalát, W. Wilczyński, *I*-convergence, *Real Anal. Exchange*; 26, 669–686, (2000-2001).

[14] V. A. Khan, Q. M. Danish Lohani, Statistically pre-Cauchy sequences and Orlicz functions, *Southeast Asian Bull. Math*.; 31, 1107-1112, (2007).

[15] P. Kumar, V. Kumar, S. S. Bhatia, Multiple sequence of Fuzzy numbers and their statistical convergence, *Mathematical Sciences, Springer*, 6 (2), 1-7, (2012).

[16] J. S. Kwon, On statistical and *p*-Cesáro convergence of fuzzy numbers, *Korean J. Comput. Appl. Math*.; 7, 195–203, (2000).

[17] I. J. Maddox, A tauberian condition for statistical convergence, *Math. Proc. Camb. PhilSoc*.; 106, 272-280, (1989).

[18] M. Matloka, Sequences of fuzzy numbers, *BUSEFAL*; 28, 28-37, (1986).

[19] F. Moričz, Statistical convergence of multiple sequences. *Arch. Math*.; 81, 82–89, (2003).

[20] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets and Systems; 33, 123-126, (1989).

[21] M. Nath, S. Roy, Some new classes of ideal convergent difference multiple sequences of fuzzy real numbers, *Journal of Intelligent and Fuzzy systems*; 31 (3), 1579-1584, (2016).

[22] F, Nuray, E. Savas, Statistical convergence of sequences of fuzzy numbers. *Math. Slovaca*; 45, 269–273, (1995).

[23] A. Şahiner, M. Gürdal, F. K. Düden, Triple sequences and their statistical convergence, Seluk J. Appl. Math; 8 (2), 49-55, (2007).

[24] A. Sahiner, B. C. Tripathy, Some *I* -related Properties of Triple Sequences, *Selcuk J. Appl. Math*.; 9 (2), 9-18, (2008).

[25] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca*; 30, 139-150, (1980).

[26] T. Šalát, B. C. Tripathy, M. Ziman, On some properties of *I*-convergence, *Tatra Mt. Math. Publ*.; 28, 279–286, (2004).

[27] E. Savas, On statistically convergent sequences of fuzzy numbers, *Inform. Sci*.; 137 (1-4), 277-282, (2001).

[28] E. Savas, On strongly-summable sequences of fuzzy numbers, *Information Sciences*; 125, 181-186, (2000).

[29] E. Savas, P. Das, A generalized statistical convergence via ideals, *Applied Mathematics Letters*; 24, 826-830, (2011).

[30] E. Savas, A. Esi, Statistical convergence of triple sequences on probabilistic normed space, Annals of the University of Craiova*, Mathematics and Computer Science Series*; 39 (2), 226 -236, (2012).

[31] M Sen, S. Roy, Some I-convergent double classes of sequences of Fuzzy numbers defined by Orlicz functions, Thai Journal of Mathematics; 11 (1), 111–120, (2013),

[32] B. C. Tripathy, A. J. Dutta, Statistically convergence triple sequence spaces defined by Orlicz function*, Journal of Mathematical Analysis*; 4 (2), 16-22, (2013).

[33] B. C. Tripathy, R. Goswami, On triple difference sequences of real numbers in probabilistic normed spaces, *Proyecciones Journal of Mathematics*; 33 (2), 157-174, (2014).

[34] B. C. Tripathy, B. Sarma, Statistically convergent double sequence spaces defined by Orlicz function, *Soochow Journal of Mathematics*; 32 (2), 211-221, (2006).

[35] B. C. Tripathy, M. Sen, On generalized statistically convergent sequences, *Indian Jour. Pure Appl. Math.;* 32 (11), 1689-1694, (2001).

[36] B. K. Tripathy, B. C. Tripathy, On *I*-convergence of double sequences, *Soochow Journal of Mathematics*; 31 (4), 549–560, (2005).

[37] L. A. Zadeh, Fuzzy sets, *Information and Control*; 8, 338-353, (1965).