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Introduction to Cartesian Geometry and Cartesianization of Complex Shapes

Nourddin Saidou

Departement of Mathematics, INSA Euromed University, Fès, Morocco

Email address:
n.saidou@insa.ueurmed.org

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Abstract: The Cartesian word or "Cartesianity" was born with the philosophy of Descart (1596 - 1650). He was at the base of a doctrine based on rationalism, that it is means the search for truth by reason. Among others, Sigmend Freud had also approached this notion of psychological point to study the enigma of thoughts in humans. Other aspects of the Cartesian word have been used in mathematical geometry, namely cartesian coordinates and Cartesian referentials. As you know, studying a shape with curved and enclosed borders is more complicated than working on shapes with linear borders without curvature. In the way, we will introduce to the Cartesian geometry and characterize he Cartesian shapes.

Keywords: Cartesian Shapes, Polytopes, Banach Spaces, Convex Sets

1. Introduction

The main of this paper is to give the theoretical foundations and later in future papers, we will give the mathematical and numerical techniques to "Cartesianize" a complex form. Among the various aspects of mathematics that have contributed to functional analysis, we must underline, the convex analysis, a branch that has had fertile tracks in analytical and even geometric forms. It will be shown that convex analysis and Cartesian analysis coincide in particular cases but generally they are different. We have to denote that this geometry will be a generalization of the polyhedron and polytopes in affine spaces [5-15]. The regular polytopes and polyhedron are convex sets but Cartesian sets are the union not necessary convex of many polytopes [16-18] (See Figure 3 below). After defining and after characterizing this new notion, you will realize how this track would open interesting doors in the field of the applications of mathematics for the benefit of applied sciences, like medicine, physics, engineering, economics and even sociology. We will give an original definition of a Cartesian set in a topological space that could be R, R^2, R^n or of infinite dimension, as well as some characterizations.

It is always difficult to study a phenomenon within an enclosed region or region with an obliquely indented contour, which is due to the problems of irregularities in its boundary. To do this, the main objective of this theory is to find a subset very close which approximate the region with a contour formed of the linear parts as segments (in R^2), or hyperplans in R^n. This is to have a similar profile of the complex region in which the phenomenon would be easier to control and study. (See Figure 1)

Figure 1. Complex region in the left and its Cartesian profile in the right.

2. Definitions and Preliminaries

Let E be a Banach space, E^j its dual, and <. , .> the dual product.

H is called an hyperplan in E if and only if, there exists a linear form x^j belonging to the dual E^j such that: $H = \{y \in E, <x^j, y>= \alpha, \alpha \in R\}$.

1). D={y ∈ E, <xj, y > ≤ α}

2). D={y ∈ E, <xj, y > ≥ α}

er and upper half spaces defined by x^j.

A set C in E is said to be con.

2.1. Remark

It is easy to prove that H, D are closed convex sets in E.

2.2. Definition

C is said to be a regular Cartesian set of E if and only if, C is the intersection of a finite family of D_i, with D_i are half-spaces of E. (See Figure 2).

$C = \cap(D_i)$

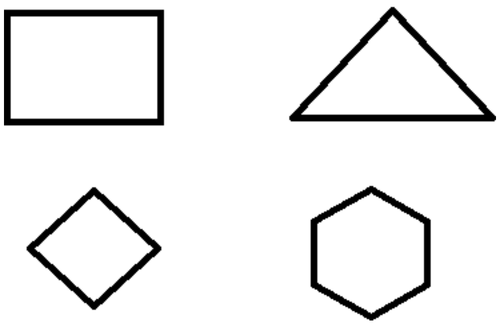

Figure 2. *Regular Cartesian Sets.*

2.3. Example

A square is a regular Cartesian set defined by the intersection of four half spaces.

[0, 1] x [0, 1] in the intersection of the following four half-spaces:

The half space below the line y = 1, the half space above y = 0, the half space to the left of the vertical axis x = 1 and the half space to the right of the vertical axis x = 0.

A triangle is a regular Cartesian set defined by the intersection of three half- spaces. (See Figure 2)

Now, we have all the necessary assets to pronounce the first definition of a Cartesian set.

In fact it would be a concatenation of regular Cartesian sets.

2.4. Definition

Let C be a set of E. We say that C is Cartesian if and only if C is a finite union of regular Cartesian sets connected to each other. (See Figure3)

$C = \cup\cap(D_{i,j})$

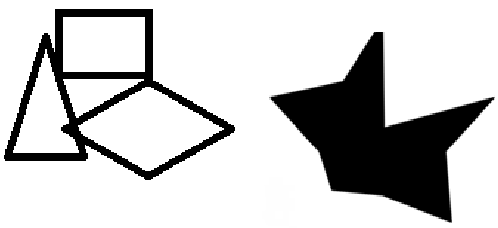

Figure 3. *Cartesian sets as a finite union of regular cartesian sets.*

2.5. Example

As illustrated in the Figure3 above, we note that set C is an union of regular Cartesian sets as it has been illustrated and defined above.

We could proceed geometrically by inverse process. That is to say, consider the Cartesian set C and subdivide it to find the regular Cartesians subsets. (See Figure 4)

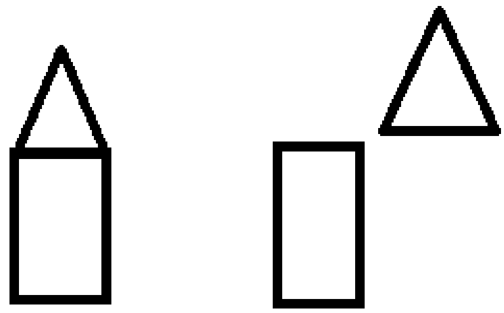

Figure 4. *A cartesian set in the left subdivided in two regular Cartesian sets in the right.*

We can extend this notion of regular Cartesian sets on R^n which are represented in the form of polyhedra [5-7] sets that have already been treated before in the context of convex analysis. Thus, Particularly if one would like to have an idea about the Cartesian set in R^n, it would be according to our definition the finite union of polyhedra. (See Figure 5).

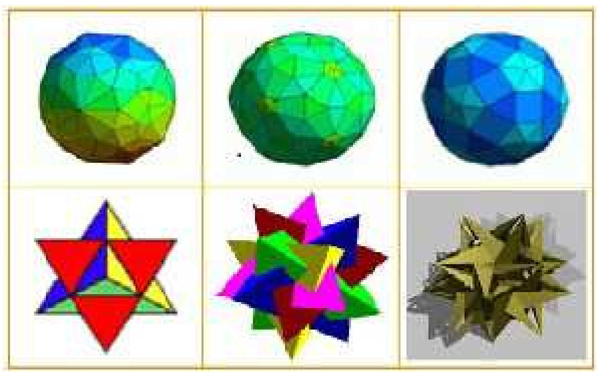

Figure 5. *Example of Cartesian sets in R^n.*

2.6. Remark

As you will remark, the notion of the boundary or contour will be the main topological property in this work because it will be the key characterization of a Cartesian set. As it was signaled above, intuitively a Cartesian set is a set whose contour is formed of lineal parts.

Thus we define below the boundary.

We will note *cl(C) respectively int(C)* the closure respectively the interior of *C*.

2.7. Definition

The topological boundary of a set C is its closure private of its interior.

$$Fr(C) = cl(C) \setminus int(C)$$

3. Topological Characterizations

Before, we have to recall and prove some topological properties.

3.1. Proposition

The interior of a finite intersection of sets D_i is the same of the finite intersection of its interior, and the finite union of interiors of D_i is included in the interior of its unions.

$$\mathrm{int}(\cap D_i) = \cap(\mathrm{int} D_i)$$

$$\cup(\mathrm{int} D_i) \subset \mathrm{int}(\cup D_i)$$

3.2. Proof

Suppose that x is in the interior of $\cap D_i$ then there exists a r>0 such that the B(x, r) is included in each D_i, wich it is means that x is in $\mathrm{int}D_i$ for each i, then x is in $\cap(\mathrm{int}D_i)$. In the other hand if x is in $\cap(\mathrm{int}D_i)$, then x is in $\mathrm{int}D_i$ for each i. There exists $r_i>0$ such that $B(x,r_i)$ is included in D_i. We choose r = min(r_i), the B(x, r) is included in $\cap D_i$. So, x is in int($\cap D_i$). Finaly.

$$\mathrm{int}(\cap D_i) = \cap(\mathrm{int} D_i)$$

For the second inclusion. Let x in $\cup$(intDi), then there exists j such that, x in $\mathrm{int}D_j$, then there exists r >0 such that the B(x, r) is included in D_j. Thus, B(x, r) is included in $\cup(D_i)$ Hence, x is in int($\cup D_i$). Therefore.

$$\cup(\mathrm{int} D_i) \subset \mathrm{int}(\cup D_i)$$

3.3. Proposition

The boundary of a finite union of sets is included in the finite union of its boundaries,

$$Fr(\cup D_i) \subset \cup Fr(D_i)$$

The equality is not always true.

3.4. Proof

Let x in Fr($\cup D_i$). According to the definition above, Fr(C) = cl(C)\int(C), x is in cl($\cup D_i$)\int($\cup D_i$). D_i are closed then x is in $\cup(D_i)$, but x is not in int($\cup D_i$). Using the proposition3.3 above, So There exists j such that x is not in $\mathrm{int}D_j$ and x is in cl(D_j). Then, there exists j such that x is in Fr(D_j). That is, x is in $\cup$ Fr(D_i).

Consequently,

$$Fr(\cup D_i) \subset \cup Fr(D_i)$$

The reverse inclusion is not true because. Let *D1=[0, 4], D2=[3, 5]. Fr(D1)={0, 4}, Fr(D2) ={3, 5} So, Fr(D1)* $\cup$*Fr(D2) = {0, 3, 4, 5}*. But *Fr(D1* $\cup$*D2) = {0, 5}*

According to the definition of the regular Cartesian set, it would be easy to characterize the boundary of such set, knowing that it is trivial to see that these sets are closed for the topology defined by the norm. The lemma below is to show that the boundary of a regular Cartesian set is not round nowhere see example a polyhedron in R^n.

3.5. Lemma

A regular Cartesian set is always Cartesian.

3.6. Proof

The proof is trivial according to the definitions.

3.7. Lemma

Let E be a Banach space. The finite intersection and the finite union of Cartesian sets are Cartesians.

3.8. Proof

Let C_k a finite family of Cartesians sets in E, then for each k, $C_k = \cup(\cap D_{i,j,k})$), then,

$$\cup_k(C_k) = \cup_k(\cup_i(\cap_j D_{i,j,k})) = \cup_k(\cap_{i,j} D_{i,j,k})$$

So $\cup(C_k)$ is also Cartesians.

In the same, $\cap(C_k) = \cap(\cup(\cap D_{i,j,k})) = \cup(\cap D_{i,j,k})$ which is Cartesian.

4. Main Results and Theorems.

Now we will begin to give characterizations linking the other notions of convex analysis and the functional analysis in general knowing that there would be a multitude of future results and digging in this track for Cartesian sets. The usefulness of the results below is to demonstrate that the Cartesianity" and the Convexity are two completely different notions which give the utility of this field (Cartesian analysis).

4.1. Proposition

A regular Cartesian set is always closed convex but the reverse is not always true.

4.2. Proof

Let C a regular Cartesian subset in E, it is clear that C is closed because C is an intersection of a finite closed sets D_i. Now we have to demonstrate that C is convex.

Let x, y $\in$ C it suffice to prove that [x, y] is included in C. Let $\alpha \in [0, 1]$, we will prove that,

$$\alpha.x + (1 - \alpha).y \in C.$$

x $\in$ C then x $\in D_i$ for each i. Then for each i, $<x^j_i, x> \leq \alpha_i$ with $x^j_i \in E^j$

the dual of E and $\alpha_i \in R$.

In the same y $\in$ C then for each i, $<x^j_i, y> \leq \alpha_i$. We can deduce that,

$$<x^j, \alpha.x + (1 - \alpha).y> \leq \alpha_i.$$

Hence C is convex.

The reverse is not true; we consider the circle C((0, 0), 1) in R^2. The C((0, 0), 1) is convex. Suppose that C = $\cap(D_i)$. In R^2, each half space can have an equation like ax + by $\leq \alpha$.

We will verify that (0, 0) $\in D_i$ for each i.

We have (0, 1) and (−1, 0) *in C*((0, 0), 1) then (0, 1) and (−1, 0) *in* D_i for each *i,* which implies that $a \leq \alpha_i$ *and* $-a \leq \alpha_i$, therefore, $\alpha_i \geq 0$. Hence

$$a.0 + b.0 \leq \alpha_i$$

then (0, 0) is in D_i for each i, which means that $(0, 0) \in C$ absurd.

4.3. Theorem

Let C to be Cartesian, then, The boundary of C is included in the union of the Hyperplans defining the $D_{i,j}$. Precisely, it is formed of linear parts.

$$F\,r(C) \subset \cup (H_{i,j}) \cap C$$

4.4. Proof.

C is Cartesian, then there exists a finite family $D_{i,j}$ such that $C = \cup \cap (D_{i,j})$.

As proved above (see Proposition3.3).

$$Fr(C) = Fr(\cup(\cap D_{i,j})) \subset \cup Fr(\cap D_{i,j}).$$

So it suffice to prove that $\cup Fr(\cap D_{i,j}) \subset \cup (H_{i,j}) \cap C$.

Let $x \in Fr(C)$. Then, $x \in \cup Fr(\cap D_{i,j})$ Then, there exists j_0 such that,

$$x \in Fr(\cap D_{i,j0}).$$

Then $x \in cl(\cap D_{i,j0}) \setminus int(\cap D_{i,j0})$. Then for each i, $x \in (\cap D_{i,j0})$ and x is not in $int(\cap D_{i,j0})$. According to the proposition3.1, see above $int(\cap D_i) = \cap(int D_i)$, we deduce that there exists i_0 such that x is not in $int(Di_0 j_0)$.

$x \in D_{i0,j0}$ means that there exists $x^j_{i0,j0} \in E^j$ and $\alpha_{i0,j0}$ such that

$$<x^j_{i0,j0}, x> \leq \alpha_{i0,j0}.$$

x is not in $int(D_{i0,j0})$ means that

$$<x^j_{i0,j0}, x> \geq \alpha_{i0,j0}$$

Hence $x \in Hi_0 j_0$, then $x \in \cup(H_{i,j})$.

Since $Fr(C) \subset C$ (C is closed), we conclude that,

$$Fr(C) \subset \cup (H_{i,j}) \cap C.$$

The equality is not always true. We construct below an example of a Cartesian set such that. the inclusion in the other hand is not true Let $C = C_1 U C_2$. As you will remark in the Figure 6 below, when the regular Cartesian set constituting the Cartesian sets are connected in more than one point, the inclusion is not true.

Figure 6. *$\cup(H_{i,j})\cap C$ in the left and $Fr(C)$ in the right.*

The next theorem resume all the contains of this first paper. The main is to characterize the Cartesian topology with the boundaries. Thus all the work and control of Cartesian region or territories will be studied only in its boundaries.

The demonstration of the theorem below will be treated in the next paper (To appear).

4.5. Theorem

A set C is Cartesian in a Banach space E if and only if, its boundary is an union of linear parts which is means that its boundary does not have any round part.

4.6. Remark

We have to recall that an applied research axis about Cartesian analysis will be studied using all the theoretical results proved above. A numerical methods will be used in order to "cartesianize" a complex set or region in $R^3$with a round boundary. Many papers will be published in order to show the applications of this axis 'Cartesian Analysis'. In the next works about Cartesian analysis, and following the current introductions, we will treat researches of topological aspects. We will define Cartesian topology and the Cartesian spaces (These spaces will be called Saidou'spaces). After that, we will start many studies about the Cartesian functions. We will define the Cartesian functions and its epigraphs. It will be an interesting work because we will find a link between convex functions and Cartesians functions, that it will be an interesting way for optimization.

5. Conclusion

In this first work, we introduced to a new mathematics foundations for a new more generalized theory that will be called "Cartesian Analysis". At first, it is a question of defining and characterizing a Cartesian form in a topological space with an infinite dimension. At the end of these definitions, we discussed the geometric and some topological properties of such forms that are easier to use than any complex forms. To say easier to use is to say simpler to study and to control. It is not easy to find the maximal Cartesian profile of a complex form but we would approximate it by these Cartesian forms and minimizing the error.

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