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Stochastic Optimal Control Theory Applied in Finance

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Abstract: In the field of investment, depending on their structures and in order to make the best decisions that are optimal, some companies are subject to some restrictions on their assets. And generally speaking, these constraints concern assets evolving in uncertainty. This paper focuses on studying a financial continuous-time Merton optimal investment problem in the case where there is a reallocation constraint with regard to the risky asset. Under this constraint, a certain rate is fixed such that the stock asset cannot be liquidated sooner than the rate. It is a stochastic control pure investment case for a large investor who faces a discounted infinite time horizon with utility function of only wealth, subject to a risk aversion coefficient. Our main goal is to characterise an optimal trading strategy for investors expecting high returns for low risks. We propose the dynamic programming method whose value function satisfies a nonlinear partial differential equation. Under homotheticity of the value function, a reduction of dimension is used in order to reduce the original two spatial dimensions problem to one dimension in a bounded domain. Numerical approximations are used to study the dynamic programming by finite difference discretisation and the convergence between the finite and the infinite time horizon problem is presented.

Keywords: Merton Problem, Stochastic Optimal Control, Dynamic Programming, Reallocation Constraint

1. Introduction

The original Merton problem known as the Merton's portfolio problem has been one of the pioneers in continuous-time finance. The problem has been introduced to solve the question of wealth strategy allocation. It concerns the case where the investor total portfolio is composed of two assets, one with a constant known rate of return called the risk-free asset (e.g. a bank account) and one with an unknown rate of return called the risky asset (e.g. a stock). According to the uncertainty on the risky asset, the investor wants to know what is the optimal proportion of her wealth to invest in each of the assets in order to maximise her total terminal utility which is on the constant relative risk aversion (CRRA) form. In his celebrated paper [1] , Merton showed that in the case of an investment concerning only the wealth held without taking into account any costs related to the investment, a strategy for an optimal allocation is to keep a fixed proportion of the total wealth in both assets and to consume at a constant rate relative to wealth. This proportion is expressed in terms of the parameters of the problem .

After Merton introduced the problem, research has continued and several extensions have been brought to generalise the problem. Proportional transaction costs have been introduced in the work of Magill and Constantinides, and the results show the existence of a non-trading region between the buying and the selling boundaries [2]. Jiang, Li, and Yi [3] studied the case of a single risky asset with proportional transaction costs holds by an investor who faces finite time horizon. It has been showed that the optimal buying boundary is not always monotone when consumption is implied. Dai and Zhong in [4] used a Penalty method for continuous-time portfolio selection with transaction costs in the case of multiple risky assets held and showed that it is never optimal to buy any risky asset when the time is close to the time maturity. Arun worked on Merton's problem with a drawdown constraint on the consumption process [5]. Under this constraint the consumption level must be kept above a fixed proportion of the running maximum of the past value consumption. If the investor's wealth ratio to the maximum past consumption has low values, consumption is bounded to the minimal level possible without going out from the drawdown constraint. But as the value increases, consumption increases with the wealth. Optimally, consume at the higher possible level while waiting for the ratio value to attain a critical level, after which the consumption level is augmenting to a new maximum.

Many other works have been studied bringing new formulations to Merton's model. A particular case of the optimal investment problem is the one with constraint on the risky asset. In this paper, we study the case where the Merton problem is subject to a reallocation constraint on the risky asset. This constraint gives an upper bound with a parameter K on the proportion of the risky asset to transfer to the risk-free asset. For any time t, it is not allowed to liquidate the stock $Y(t)$ at a rate $u(t)$ greater than $KY(t)$. A real situation that motivates such a constraint is that sometimes for investors who allocate money to different risky assets, there are some initial constraint agreements such that the amount of money to removed from an asset is upper bounded. So the question is to know, how does the optimal investment strategy of the Merton's problem change with such a condition added to the model? We then study a problem under this constraint for an investor who cares about risk and faces an infinite time horizon. The investor wants to maximise her expected utility function U subject to a risk aversion coefficient $0 < p < 1$. The problem is first presented for infinite time horizon and then solved for finite time horizon. Unlike the work of Bakshi and Chen [6], we consider a pure investment for which the utility function is from wealth w only, we do not include consumption. The power utility function takes then the form $U(w) = w^p$. The principal objective of this work is to establish an optimal trading strategy by solving a partial differential equation, the so-called Hamilton-Jacobi-Bellman (HJB) equation associated to the problem. A numerical approximation is provided to study the solution to the HJB equation known as the value function and which provides the maximum utility. However, the dimension of the problem is an obstacle. With both investments in stocks and bank we need a two spatial dimensions which can bring complexity to numerical scheme. We then proceed to a reduction of dimension in space by homotheticity transformation of the value function. This leads to solve the problem for the infinite or finite time horizon in one spatial dimension. The reduction of dimension is justified by the fact that to compress or to extend the wealth of the two investments leads to a single compression or extension of the maximum utility. Therefore one way to obtain this reduction is to consider one of the investments as unity. By the implicit finite difference methods we compute the value function in one space dimension. For the finite time horizon, the solution converges to the infinite horizon as the terminal time T goes to infinity, and that convergence is rather quick.

The case we are studying in this paper is that of an investor who has money market (bank account) and stock as assets. The objective function of the problem is given by a constant relative risk aversion function of the total wealth held with utility risk coefficient p. We look for an optimal strategy that will help investors for risky investment decision in order to minimise the losses that may incurred from delaying a trade. Indeed, if no action of buying is made, too much money is kept and the investor loses money in terms of opportunity cost, explain Sethi and Thompson [7]. She could have earned higher returns by buying stock such as bonds. The investor is then in what is called a long position. Likewise, if no action of selling is made, too much stock is held, the money market is small and money is lost in terms of opportunity cost. The investor is said to be in a short position. Using the dynamic programming method, we compute an optimal strategy that will help such investors to exercise at the best moment in such way to benefit from any particular investment.

2. Stochastic Optimal Control Theory for Investment Strategies

Optimal control theory is developed to find optimal ways to control dynamical systems over time. This concerns generally deterministic optimal control problems for which the outcome is known, [7]. This notion finds also its usefulness in several applications other than finance. See for exemple [8] and [9]. However, Stochastic optimal control theory deals with the uncertainly in the evolution of a dynamical system considered.

The use of stochastic optimal control theory in finance is investigated in this part of the work. We show this by an application of the dynamic programming method to the problem of finding optimal investment strategies.

Risk is an important component to consider before investing. In financial modelling, we associate to the utility function of a considered problem a parameter, p, referring to the risk and called the risk aversion coefficient. The utility function is then said to be constant relative risk aversion and takes different forms according to the problem to solve. We note it by $U(c) = c^p$, where c represents the total wealth held and $0 < p < 1$.

2.1. Model Formulation

We consider the standard financial market as defined by Egriboyun and Soner in [10], Davis and Norman in [11], in which only two investments are considered: a risk-free asset "bank account" and a risky asset "stock" with respective price dynamics given by

$$\mathrm{d}P_0(t) = rP_0(t)\mathrm{d}t \tag{1}$$

$$\mathrm{d}P_1(t) = P_1(t)(\mu\mathrm{d}t + \sigma\mathrm{d}W(t)), \tag{2}$$

where $r > 0$ is the constant risk-free rate, $\mu > r$ is the constant expected risky rate and σ is the stock's volatility. The stock price evolves according to the geometric Brownian motion form. The process $\{W(t); t \geq 0\}$ is a standard one dimensional Brownian motion on a filtered probability space

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with $W(0) = 0$ almost surely. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is assumed to be right-continuous and each $\mathcal{F}_t$ contains all the $P - null$ sets of $\mathcal{F}$. The initial position of the investor is given by x_0 dollars invested in the bank account and in stock y_0 dollars invested. Let denote by L the cumulative dollar value for the purpose of buying stock (or selling money) and by M the cumulative value for selling stock. Both are right-continuous, nonnegative and nondecreasing $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process with $L(0) = M(0) = 0$.

We refer by $(X(t), Y(t))$ the position of the investor at time t where $X(t)$ represents the dollar value in bank and $Y(t)$ the dollar value in stock. The evolution of these dynamics are described by the given equations:

$$\mathrm{d}X(t) = rX(t)\mathrm{d}t + u(t)\mathrm{d}t - \mathrm{d}L(t), \quad X(0) = x_0 \tag{3}$$

$$\mathrm{d}Y(t) = -u(t)\mathrm{d}t + \mathrm{d}L(t) + \mu Y(t)\mathrm{d}t + \sigma Y(t)\mathrm{d}W(t), \quad Y(0) = y_0, \tag{4}$$

Where $u(t)$ is the rate of liquidation from stock to money market such that $\mathrm{d}M(t) = u(t)\mathrm{d}t$ and it is constrained to

$$u(t) \leq KY(t), \quad \forall t. \tag{5}$$

The money on the risky asset to be transferred to risk-free asset is bounded by a parameter K. The stock held cannot be liquidated faster than $u(t)$. At time t, the portfolio or the total wealth of the investor is given by $B(t) = X(t) + Y(t)$, then the evolution of the portfolio over time is described as follows

$$\mathrm{d}B(t) = (rX(t) + \mu Y(t))\mathrm{d}t + \sigma Y(t)\mathrm{d}W(t). \tag{6}$$

We will note $(x(t), y(t))$ to refer the position rather than employing capital letters.

Remark: As done in Egriboyun and Soner [10], we study a pure investment problem, which refers only to the wealth held. When consumption and transaction costs are considered, (3) and (4) would be represented as

$$\begin{aligned}\mathrm{d}X(t) &= (rX(t) - C(t))\mathrm{d}t - (1+\lambda)\mathrm{d}L + (1-\mu)\mathrm{d}M(t)\\ \mathrm{d}Y(t) &= \alpha Y(t)\mathrm{d}t + \sigma Y(t)\mathrm{d}W(t) + \mathrm{d}L(t) - \mathrm{d}M(t),\end{aligned}$$

where $C(t) \geq 0$ is the consumption rate, λ and μ are respectively the transaction costs incurred on purchase and sale of the stock. For more details, this problem is studied by Dai, Jiang, Li and Yi in [3] for the one dimensional and studied by Dai and Zhong [4] for higher dimensions. Before the formulation of our problem, we state the Merton's portofolio problem as an introduction to the pure problem considered in this paper since we are studying a Merton case with reallocation constraint. This is a standard investment model for continuous-time in finance. We present the problem without taking into account the consumption.

2.2. Merton Problem

The problem is formulated as Touzi [12]. The Merton problem presented here is concerning an investor who must allocate her total wealth or fortune between a risky asset or a risk-free asset in order to maximise the expected utility of her terminal wealth. Let denote $X(t)$ the total wealth at time $t \in [0, T]$ and X_0 the initial wealth. We note $\pi(t)$ the proportion at time t of the wealth to invest in the risky asset and the remaining fraction $1 - \pi(t)$ to invest in the risk-free asset. The wealth evolves according to the stochastic differential equation

$$\mathrm{d}X(t) = X(t)[(r + (\mu - r)\pi(t))\mathrm{d}t + \pi(t)\sigma\mathrm{d}W(t)], \tag{7}$$

with initial condition $X_0 = x$ and where r represents the rate of return on the risk-free asset, μ and σ are respectively the expected return and the volatility on the risky asset, and $W(t)$ is the Wiener Brownian motion. The value function is given as

$$V(x,t) = \sup\Big(E(u(X(T)))\Big), \tag{8}$$

where the utility function $u(x)$ follows the constant relative risk aversion of the form $u(x) = x^p$. The associated Hamilton-Jacobi-Bellman equation is represented as follows

$$\varphi_t(x,t) + \sup_\pi\Big((r + (\mu - r)\pi)x\varphi_x(x,t) + \frac{1}{2}\sigma^2\pi^2x^2\varphi_{xx}(x,t)\Big) = 0, \tag{9}$$

Where $\varphi(x,t)$ is a twice continuously differentiable function on the state process x and the time t. By verification theorem, the function $\varphi(x,t)$ corresponds to the value function $V(x,t)$. Since the utility function is of the form x^p, the value function can be defined as $V(x,t) = x^pV(1,t)$, and we set a function $h(t) = V(1,t)$. Plugging this into (9), the result is an ordinary differential equation on h given as follows

$$h_t + (ph)\sup_\pi\Big(r + (\mu - r)\pi + \frac{1}{2}(p-1)\sigma^2\pi^2\Big) = 0. \tag{10}$$

Solving this equation for π, the maximiser is given by

$$\pi = \frac{\mu - r}{\sigma^2(1-p)}, \tag{11}$$

and (10) is now reduced to

$$h_t + h\Big[rp + \frac{1}{2}\frac{(\mu - r)^2p}{\sigma^2(1-p)}\Big] = 0. \tag{12}$$

Finally solving (12) for h we get the result

$$h(t) = \exp\left[rp + \frac{1}{2}\frac{(\mu - r)^2 p}{\sigma^2(1-p)}\right](T - t). \tag{13}$$

Therefore the value function $V(x,t) = x^p h(t)$ satisfies the HJB equation and the optimal control portofolio allocation for the Merton problem is given by $\pi^*(t) = \pi$. The optimal strategy is then to allocate the constant proportion π in the risky asset, and the constant proportion $1 - \pi$ in the risk-free asset. For more details about the Merton original problem see also Holth [13]. This result will be useful to set the solution of our problem in the next section.

2.3. Optimisation Problem

A problem of minimising losses can be reformulated as a one of maximising winnings on the wealth held. Our problem then becomes to find an optimal strategy for maximising the expected utility obtained from the total wealth held. We first set the problem in infinite time horizon and then solve for the finite time horizon. In the following we will see that the solution in the two cases converge.

The objective function related to our problem is in the Lagrange form with infinite time horizon given by the follows

$$J(x,y) = E\left[\int_0^\infty e^{-\beta\tau}(x(\tau) + y(\tau))^p dt\right], \tag{14}$$

where $\beta > 0$ is the discount rate, $0 < p < 1$ is the risk aversion coefficient and, $(x(\tau) + y(\tau))$ represents the total wealth at time τ. The solvency set is defined as

$$\mathcal{S} = \{(x,y) : x \geq 0, y \geq 0\}, \tag{15}$$

which is the closed set of positions such that the investor will always hold a positive wealth by admissible strategies, Egriboyun and Soner in [10]. Stock and bank are animated by purchases and sales of assets. We sell shares of stock to satisfy the money requirements, similarly we sell money to "refuel" the stock. For our case, the controls variables will then be given by (L, M). The maximum cost will be obtained from the value function associated to (14) and defined for all $(x,y) \in \mathcal{S}$ as follows

$$V(x,y) = \sup_{L,M} E\left[\int_0^\infty e^{-\beta\tau}(x(\tau) + y(\tau))^p dt\right]. \tag{16}$$

As Egriboyun and Soner [10], an investment policy (L, M), with $L(0) = M(0) = 0$ is said admissible for the initial position (x_0, y_0) if position $(x(t), y(t))$ which is solution of (3) and (4) is in $\mathcal{S}$ for all $t \geq 0$ and we note by $\mathcal{A}(x_0, y_0)$ the set of all such policies.

To find the optimal policy (L, M), we need to find the partial differential HJB equation that the value function $V(x,y)$ must satisfy. From [10], the function $V(x,y)$ satisfy the following HJB equation:

$$\min\{\mathcal{L}\varphi + Ky(\varphi_y - \varphi_x) - (x+y)^p; (\varphi_x - \varphi_y)\} = 0, \tag{17}$$

With $(x,y) \in \mathcal{S}$ and where $\mathcal{L}\varphi(x,y) = \beta\varphi(x,y) - \frac{1}{2}\sigma^2 y^2 \varphi_{yy}(x,y) - \mu y \varphi_y(x,y) - rx\varphi_x(x,y)$ is a second order differential operator.

To facilitate the computation of the value function, [10] propose a reduction of dimension by using homothetic transformation of the value function. This is done since the compression or extension of wealth invested results in a simple compression or extension of the maximum utility. Therefore, one of the investment can be considered as unity.

The notion of homotheticity of the value function $V(x,y)$ is then applied, defined as

$$V(\gamma x, \gamma y) = \gamma^p V(x,y), \tag{18}$$

which means that if all arguments of the function are multiplied by a factor, then its value is multiplied by some power of this factor. We note that the factor γ will represent in our case the total wealth.

Using this definition, we reduce our problem to one variable as the following

$$g(z) = V(1,z). \tag{19}$$

We solve the HJB equation for $g(z)$ and then use the homothety property to recover the value of $V(x,y)$ as

$$V(x,y) = x^p V(1, \frac{y}{x}) = x^p g\left(\frac{y}{x}\right). \tag{20}$$

According to (20) and the definition of homotheticity of the value function, the variable z in (19) is in fact $z = \frac{y}{x+y}$ which represents the stock ratio and $1 - z = \frac{x}{x+y}$ will represent the bank account ratio. We can now set a function of the stock and the money bank as function of one variable z, this is $f(z) = V(1-z, z)$. The value function in (20) becomes

$$V(x,y) = (x+y)^p f\left(\frac{y}{x+y}\right). \tag{21}$$

By a verification theorem, $\varphi(x,y) = V(x,y)$. Using the value function as given in (21), we calculate the respective derivatives as appearing in the expression of the operator $\mathcal{L}\varphi(x,y)$. We have:

$$V_x = \left(\frac{\partial}{\partial x}(x+y)^p\right) f\left(\frac{y}{x+y}\right) + (x+y)^p \left(\frac{\partial}{\partial x} f\left(\frac{y}{x+y}\right)\right) = p(x+y)^{p-1} f\left(\frac{y}{x+y}\right) - y(x+y)^{p-2}\frac{\partial f}{\partial x}\left(\frac{y}{x+y}\right).$$

Since $z = \frac{y}{x+y}$, and to simplify we use $f\left(\dfrac{y}{x+y}\right) = f$,

$$V_x = p(x+y)^{p-1} f - z(x+y)^{p-1} f_z = pf - zf_z.$$

Similarly we calculate

$$V_y = \left(\frac{\partial}{\partial y}(x+y)^p\right) f\left(\frac{y}{x+y}\right) + (x+y)^p \left(\frac{\partial}{\partial y} f\left(\frac{y}{x+y}\right)\right).$$

Using $\dfrac{x}{x+y} = 1 - z$,

$$V_y = p(x+y)^{p-1} f + (x+y)^{p-1}(1-z) f_z = pf + (1-z) f_z.$$

And from V_y we get

$$\begin{aligned} V_{yy} &= p(p-1)(x+y)^{p-2} f + p(x+y)^{p-1} \frac{x}{(x+y)^2}\left(\frac{\partial}{\partial y} f\left(\frac{y}{x+y}\right)\right) - \frac{x}{(x+y)^2}(x+y)^{p-1}\left(\frac{\partial}{\partial y} f\left(\frac{y}{x+y}\right)\right) \\ &+ (1-z)(p-1)(x+y)^{p-2}\left(\frac{\partial}{\partial y} f\left(\frac{y}{x+y}\right)\right) + (1-z)(x+y)^{p-1}\frac{x}{(x+y)^2}\left(\frac{\partial^2}{\partial y^2} f\left(\frac{y}{x+y}\right)\right) \\ &= p(p-1)(x+y)^{p-2} f + (p(1-z)(x+y)^{p-2} - (1-z)(x+y)^{p-2} \\ &+ (1-z)(p-1)(x+y)^{p-2}) f_z + (1-z)^2 (x+y)^{p-2} f_{zz} \\ &= p(p-1) f + 2(p-1)(1-z) f_z + (1-z)^2 f_{zz}. \end{aligned}$$

We take the expressions found for V_x, V_y and V_{yy} in the HJB equation given in (17) and take $y = z$, $x = 1 - z$ to get

$$\begin{aligned} \min\Bigg(&(\beta + \frac{1}{2}\sigma^2 p(1-p) z^2 - \mu p z - rp(1-z)) f \\ &+ (-\mu z(1-z) + \sigma^2 z^2 (1-p)(1-z) + Kz + \\ &rz(1-z)) f_z - \frac{1}{2}\sigma^2 z^2 (1-z)^2 f_{zz} - 1;\ -f_z \Bigg) = 0. \end{aligned} \tag{22}$$

If $V_x(x, y) - V_y(x, y) = 0$ at some position (x, y), then this holds at all positions through (x, y). A suggestion is that the limit between the selling stock region and the selling money market region is a line through the origin. This is called the optimal line.

Now we describe the HJB equation for the finite horizon problem. We note that the only difference with the problem in infinite horizon is the factor time. For this case the time is considered as parameter. The HJB equation associated to this problem will then contain a term derivative in time φ_t and it is given by

$$\min\{\varphi_t + \mathcal{L}\varphi + Ky(\varphi_y - \varphi_x) - (x+y)^p; (\varphi_x - \varphi_y)\} = 0, \tag{23}$$

for $(x, y, t) > 0$ and where $\mathcal{L}\varphi$ is defined as for the infinite horizon case. The value function is a function of three variables, we have $V(x, y, t)$. The reduction of dimensions as done in (21) leads in this case to the following homotheticity transformation

$$V(x, y, t) = (x+y)^p f\left(\frac{y}{x+y}, t\right), \tag{24}$$

with $f(z, t) = V(1 - z, z, t)$.

The expressions of V_x, V_y and V_{yy} will contain the term f_t, plugging them into (23), we obtain

$$\begin{aligned} \min\Bigg(&f_t + (\beta + \frac{1}{2}\sigma^2 p(1-p) z^2 - \mu p z - rp(1-z)) f + (-\mu z(1-z) + \sigma^2 z^2 (1-p)(1-z) + Kz + \\ &rz(1-z)) f_z - \frac{1}{2}\sigma^2 z^2 (1-z)^2 f_{zz} - 1;\ -f_z \Bigg) = 0. \end{aligned} \tag{25}$$

From (25), the result to the original Merton problem presented in Section (2.2) with utility function as in our case, $(x+y)^p$, gives the following HJB equation

$$\beta V(z) - rzV_z(z) - \sup_{\pi}\left(\pi(\mu - r)zV_z(z) + \frac{1}{2}\pi^2\sigma^2 z^2 V_{zz}(z) - z^p\right) = 0. \tag{26}$$

The value function solving this PDE is given by

$$V(x, y) = V^{mer}(x + y)^p, \tag{27}$$

where

$$V^{mer} = \left(\beta - rp - \frac{1}{2}\frac{p(\mu - r)^2}{\sigma^2(1 - p)}\right),^{-1} \tag{28}$$

with constraint on the discount rate as $\beta > rp + \frac{1}{2}\frac{p(\mu - r)^2}{\sigma^2(1 - p)}$ to guarantee a solution to the problem. The optimal control maximiser is given by

$$\pi^* = \frac{\mu - r}{\sigma^2(1 - p)}. \tag{29}$$

For more details about those results see Soner, [14]. To compute the value function $V(x, y, t)$ we compute the function $f\left(\frac{y}{x + y}, t\right)$ that we will compare to V^{mer} of the Merton problem since both act as coefficient of the utility $(x + y)^p$. We then call $f(z, t)$ the coefficient function. The next step of the work is to compute the coefficient function and the optimal strategy associated using numerical techniques.

3. Numerical Approximations and Results

Optimal control problems are usually nonlinear and do not have analytical solutions. To solve the problem stated in Section 2.3 which consists in finding the value function associated to the nonlinear partial differential equation or the HJB equation, we then use numerical approximation methods. We use finite difference methods to approximate numerically the value function. Finite difference methods are efficient for solving partial differential equations. We deal with the central finite difference method to approximate the partial derivatives with respect to space. Three methods can be used to approximate the partial derivative with respect to time, the implicit, explicit and Crank-Nicolson difference. To solve our problem we use the implicit method.

3.1. Finite Difference Methods

The value function $V(x, y, t)$ described in (24) solves the PDE in Equation (23) if the function $f(z, t) = V(1 - z, z, t)$ solves the PDE in (25). In fact, it is the function $f(z, t)$ that is computed in order to compute the value function. Doing this by the finite difference methods involves to solve the PDE on a discrete space-time grid (z, t), with mesh steps $(\Delta z, \Delta t)$ for the space and the time respectively. Since the reduction of dimension done in (20) creates an infinite domain $[0, \infty)$ for $f(z, t)$, we truncate the unbounded domain to the bounded domain [0,1] to allow the computation by computer. We discretise the time period interval [0,T] into M subintervals with terminal time T, and the space interval [0,Z] into N subintervals with terminal value $Z = 1$, such that

$$\begin{aligned} Z &= 0, \Delta z, 2\Delta z, \ldots, N\Delta z \\ T &= 0, \Delta t, 2\Delta t, \ldots, M\Delta t, \end{aligned} \tag{30}$$

and $\Delta z = \frac{Z}{N}$, $\Delta t = \frac{T}{M}$. The function f can be denoted in grid form at position (z_i, t_j) as $f_i^j = f(i\Delta z, j\Delta t)$ with $i = 0, \ldots, N$ and $j = 0, \ldots, M$. Figure 1 below shows the grid discretisation.

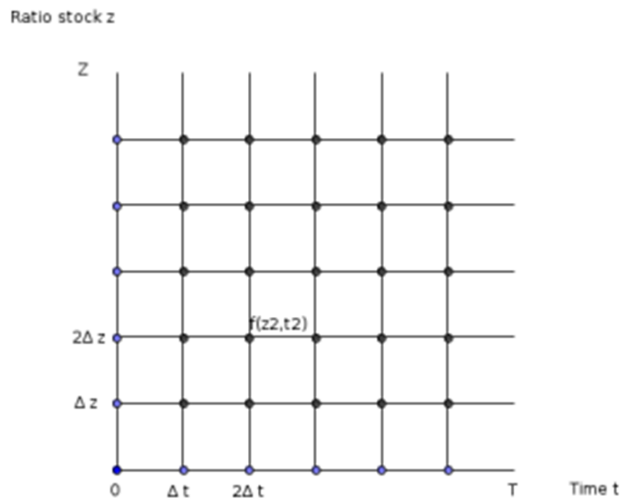


Figure 1. *Space-time discretisation.*

From Umeorah in [15], the following finite difference methods can be used for the approximation of the coefficient function $f(z, t)$:

(1) Forward difference in time and in the underlying stock respectively. We know the position (z_i, t_j) and look for the position (z_{i+1}, t_{j+1})

$$\begin{aligned} \frac{\partial f}{\partial t} &\approx \frac{f_i^{j+1} - f_i^j}{\Delta t} \\ \frac{\partial f}{\partial z} &\approx \frac{f_{i+1}^j - f_i^j}{\Delta z}. \end{aligned} \tag{31}$$

(2) Backward difference in time and in the underlying stock respectively. We know the position (z_{i+1}, t_{j+1}) and look for the position (z_i, t_j)

$$\begin{aligned} \frac{\partial f}{\partial t} &\approx \frac{f_i^j - f_i^{j-1}}{\Delta t} \\ \frac{\partial f}{\partial z} &\approx \frac{f_i^j - f_{i-1}^j}{\Delta z}. \end{aligned} \tag{32}$$

(3) Central difference in time and in the underlying stock

respectively.

$$\frac{\partial f}{\partial t} \approx \frac{f_i^{j+1} - f_i^{j-1}}{2\Delta t}$$
$$\frac{\partial f}{\partial z} \approx \frac{f_{i+1}^{j} - f_{i-1}^{j}}{2\Delta z}. \tag{33}$$

(4) Second derivative with respect to the underlying stock

$$\frac{\partial^2 f}{\partial z^2} \approx \frac{f_{i+1}^{j} - 2f_i^j + f_{i-1}^{j}}{(\Delta z)^2}. \tag{34}$$

3.2. Implicit Finite Difference Method

Recall from (25) the PDE to solve:

$$\min\Big(f_t + (\beta + \frac{1}{2}\sigma^2 p(1-p)z^2 - \mu p z - rp(1-z))f$$
$$+ (-\mu z(1-z) + \sigma^2 z^2 (1-p)(1-z) + Kz + rz(1-z))f_z$$
$$- \frac{1}{2}\sigma^2 z^2 (1-z)^2 f_{zz} - 1;\ -f_z \Big) = 0. \tag{35}$$

The implicit finite difference method considers forward difference in time, therefore the discretisations are computed at time $(j+1)\Delta t$. For the first derivative of space it considers backward difference in space and for the second derivative it considers the standard approximation. The numerical approximations are as follows: The approximation of the first derivative in time is

$$\frac{\partial f}{\partial t} \approx \frac{f_i^{j+1} - f_i^{j}}{\Delta t}, \tag{36}$$

and in space we have

$$\frac{\partial f}{\partial z} \approx \frac{f_i^{j+1} - f_{i-1}^{j+1}}{\Delta z}. \tag{37}$$

For the second derivative in space we get

$$\frac{\partial^2 f}{\partial z^2} \approx \frac{f_{i+1}^{j+1} - 2f_i^{j+1} + f_{i-1}^{j+1}}{(\Delta z)^2}. \tag{38}$$

Substituting these approximations into (35) we get

$$B_i f_{i-1}^{j+1} + A_i f_i^{j+1} + C_i f_{i+1}^{j+1} = f_i^j + \Delta t \tag{39}$$

where

$$\begin{aligned}
A_i &= 1 + \frac{\Delta t}{(\Delta z_i)^2}\sigma^2 z_i^2 (1-z_i)^2 + \frac{\Delta t}{\Delta z_i}\Big[-\mu z_i(1-z_i) + \sigma^2 z_i^2 (1-p)(1-z_i) + Kz_i + r(1-z_i)z_i \Big] \\
&\quad + \Delta t \Big[\beta + \frac{1}{2}\sigma^2 p(1-p)z_i^2 - \mu p z_i - r(1-z_i)p \Big], \\
B_i &= \frac{\Delta t}{\Delta z_i}\Big[\mu z_i(1-z_i) - \sigma^2 z_i^2 (1-p)(1-z_i) - Kz_i - r(1-z_i)z_i \Big] + \frac{\Delta t}{(\Delta z_i)^2}\Big[-\frac{1}{2}\sigma^2 z_i^2 (1-z_i)^2 \Big], \\
C_i &= \frac{\Delta t}{(\Delta z_i)^2}\Big[-\frac{1}{2}\sigma^2 z_i^2 (1-z_i)^2 \Big].
\end{aligned}$$

When we expand (39) we have the following

$$\begin{aligned}
B_0 f_{-1}^{j+1} + A_0 f_0^{j+1} + C_0 f_1^{j+1} &= f_0^j + \Delta t \\
B_1 f_0^{j+1} + A_1 f_1^{j+1} + C_1 f_2^{j+1} &= f_1^j + \Delta t \\
B_2 f_1^{j+1} + A_2 f_2^{j+1} + C_2 f_3^{j+1} &= f_2^j + \Delta t \\
\vdots \qquad\qquad &\qquad \vdots \\
B_{N-1} f_{N-2}^{j+1} + A_{N-1} f_{N-1}^{j+1} + C_{N-1} f_N^{j+1} &= f_{N-1}^j + \Delta t \\
B_N f_{N-1}^{j+1} + A_N f_N^{j+1} + C_N f_{N+1}^{j+1} &= f_N^j + \Delta t.
\end{aligned}$$

We can present this in a matrix form as

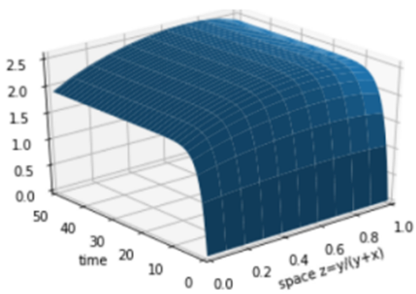
$$\begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & & & & \\ & & & B_{N-1} & A_{N-1} & C_{N-1} \\ & & & & B_N & A_N \end{bmatrix} \times \begin{bmatrix} f_0^{j+1} \\ f_1^{j+1} \\ f_2^{j+1} \\ \\ f_{N-1}^{j+1} \\ f_N^{j+1} \end{bmatrix} = \begin{bmatrix} f_0^j + \Delta t \\ f_1^j + \Delta t \\ f_2^j + \Delta t \\ \\ f_{N-1}^j + \Delta t \\ f_N^j + \Delta t \end{bmatrix} \tag{40}$$

This is the tridiagonal system to compute in order to find the numerical solution to the PDE HJB equation. At the boundary point $z = 0$ the term $B_i f_{i-1}^{j+1}$ is identically equal to zero for $i = 0$ and similarly at point $z = N$, the term $C_i f_{i+1}^{j+1}$ is identically equal to zero for $i = N$.

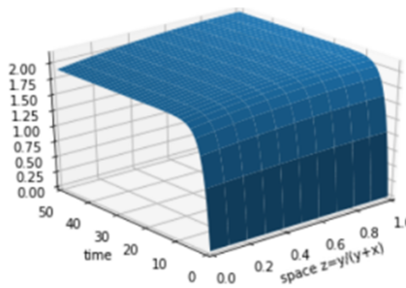
3.3. Numerical Results

From [10], the following parameters are used in the model, μ=15%, r=5%, p=0.5, β=0.5 and σ=100%. The Merton proportion given by the maximiser in (29) is obtained as π^{mer}=0.2 and gives the optimal value for the amount to hold in stock and for this value, the factor V^{mer} gives the optimal value function. We note that the two factors V^{mer} and π^{mer} depend only on the parameters and not on the state pair (x, y). This means that a constant fraction of the wealth is invested in stock. But the coefficient function $f(z,t)$ depends on (x, y) through the ratio stock $z = \dfrac{y}{x+y}$. We can then set an optimal investment strategy such that the ratio stock in our constrained problem must be always less than π^{mer} and keep increasing as the factor constraint K gets larger. The case $K = \infty$ gives the optimality for the infinite horizon problem. When the ratio z is exceeded by the optimal value, the money market can always be sold and stock can be bought at any rate to reach the optimality and the coefficient function $f(z,t) \approx V^{mer}$. But when the risky ratio z exceeds the optimal value, we are not allowed to sell stock at any rate greater than Ky as given by the Constraint in (5), and the coefficient function is far from the value V^{mer}. The investor is then losing.

By numerical analysis using Python software, Figure 2a and Figure 2b below show the coefficient function $f(z,t)$ in space-time [0,1] $\times$ [0,50] for value of K equal 1 and 3 respectively.



(a) *Parameter K=1*

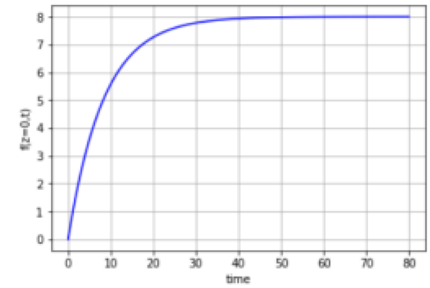


(b) *Parameter K=3*

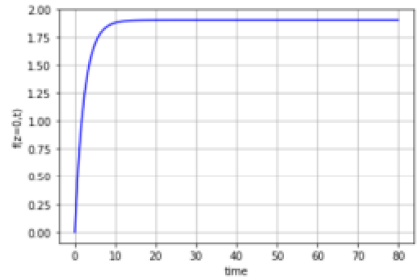
Figure 2. *Coefficient function for different values of K.*

The solution to infinite time horizon shows quicker convergence. For the same set of parameters as above, with time interval [0,80], the numerical approach for the derivation with respect to time becomes zero at time t $\approx$20. The convergence depends on the value of the discounting factor β since the factor integrand under the objective function is of $e^{-\beta t}$ and $(x(t) + y(t))^p$.

Figure 3a and Figure 3b below show the speed of convergence for different values of β.



(a) *Factor β=0.1*



(b) *Factor β=0.5*

Figure 3. *Convergence to infinite horizon solution at z=0.*

4. Conclusion

An investment Merton's optimal problem with a reallocation constraint to the stock has been studied in this paper. Under this constraint, for a pure investment problem as defined in Section 2, we have observed that the optimal investment strategy to adopt in order to maximise the objective function defined in (14) is to invest a constant proportion of the wealth in the risky asset, the stock, as in the Merton's investment problem. In other words, it is optimal to stay on the Merton's line which passes through the origin position (x, y). But, since the financial market is subject to changes (thus this problem studied by stochastic optimal control theory), the investor might not stay on the optimal line. So, when we are not on the line, if the stock held exceeds the optimal proportion given by π^* in (29) at time t, we sell the stock at the rate $Ky(t)$ which is maximum, and invest in the money market; if the money market is in excess, we sell money by buying stock. We studied numerically the dynamic programming method used to solve our problem and noticed that for large enough values of the parameter K, our solution given by the value function is not too far from the optimal value function for the Merton's problem; the convergence is quick, and for those values of K, the optimal proportion in stock is increased. We also observed that the finite time horizon problem converges rather quickly to the infinite time horizon since from a certain time around t=20, the derivative term with respect to time goes to zero and the convergence is quick for large values of the discounting factor β. By a reduction of the spatial dimension, using finite difference methods, we presented some illustrations describing all those observations.

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